Dualities in Convex Algebraic Geometry

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Philipp Rostalski

joint work with B. Sturmfels

Department of Mathematics
UC Berkeley

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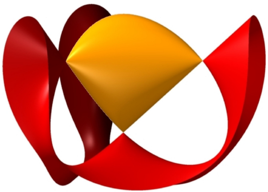
What is Convex Algebraic Geometry?

Convex geometry

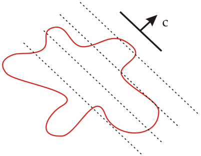
Convex body: P

Algebraic geometry



Algebraic variety: X

Optimization



Optimization problem: $\text{Minimize}_{x \in S} \; c^T x$

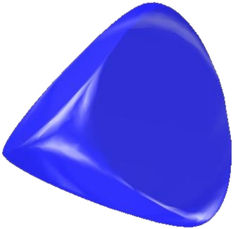
Notions of duality: Convex geometry

Let P be a full-dimensional convex body in V and assume that $0 \in \mathrm{int}(P)$.

Def. Dual convex body:

$$P^{\Delta} = \left\{ u \in V^* \mid \forall x \in P : u^T x \leq 1 \right\}$$

Convex body P



Dual body P^{Δ}

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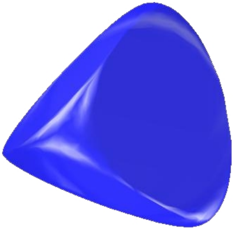
Convex body P



Dual body P^{Δ}

Important: $(P^{\Delta})^{\Delta} = P$.

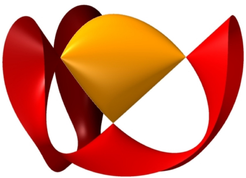
Notions of duality: Algebraic geometry

Let $X \subset \mathbb{P}^n$ be a projective variety and $u = (u_0 : u_1 : \cdots : u_n) \in (\mathbb{P}^n)^\vee$ represents the hyperplane $\{x \in \mathbb{P}^n \mid \sum_i u_i x_i = 0\}$.

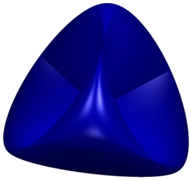
Def. Dual variety:

The *dual variety* X^* of an algebraic variety X is the closure of

$$\{u \in (\mathbb{P}^n)^\vee \mid \ u \text{ is tangent to } X \text{ at } x \in X_{\text{reg}}\}.$$



Algebraic varieties X



Dual varieties X^*

Example: Dual varieties

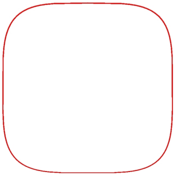
Trivial example: $L^* = L^\perp$ for any linear subspace $L \subset V$.

Example: Dual varieties

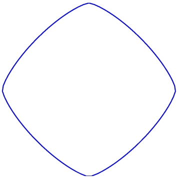
Trivial example: $L^* = L^\perp$ for any linear subspace $L \subset V$.

Example: The variety $X = V(x^4 + y^4 - z^4) \subset \mathbb{P}^2$ yields

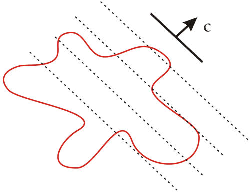
$$X^* = V\big(a^{12} + 3a^8b^4 + 3a^4b^8 + b^{12} - 3a^8c^4 + 21a^4b^4c^4 - 3b^8c^4 + 3a^4c^8 + 3b^4c^8 - c^{12}\big).$$



Algebraic variety X



Dual variety X^*



Let

$$\begin{array}{lll} \underset{x}{\text{Minimize}} & c^T x & \\ \text{subject to} & g_i(x) \leq 0, & i = 1, \ldots, m \\ & h_j(x) = 0, & j = 1, \ldots, p \end{array} \tag{1}$$

be a constrained optimization problem and

$$\begin{array}{rcl} L : \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p & \rightarrow & \mathbb{R}^n \\ (x, \lambda, \mu) & \mapsto & c^T x + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^p \mu_j h_j(x) \end{array}$$

the associated Lagrangian.

Then optimization problem (1) can be written as

$$u^* \;=\; \underset{x\in\mathbb{R}^n}{\text{Minimize}}\;\underset{\mu\in\mathbb{R}^p\text{ and }\lambda\geq 0}{\text{Maximize}}\quad L(x,\lambda,\mu)$$

and the *dual optimization problem* to (1) is

$$v^* \;=\; \underset{\mu\in\mathbb{R}^p\text{ and }\lambda\geq 0}{\text{Maximize}}\;\underbrace{\underset{x\in\mathbb{R}^n}{\text{Minimize}}\quad L(x,\lambda,\mu)}_{\phi(\lambda,\mu)}\,.$$

Optimality conditions [Karush,Kuhn and Tucker]

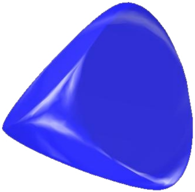
Let (x, λ, μ) be primal and dual optimal solutions with $u^* = v^*$ (strong duality). Then

$$c + \sum_{i=1}^{m} \lambda_i \cdot \nabla_x g_i\Big|_x + \sum_{j=1}^{p} \mu_j \cdot \nabla_x h_j\Big|_x = 0, \quad (2)$$

$$\begin{aligned} g_i(x) &\leq 0 \quad \text{for } i = 1, \ldots, m, \\ \lambda_i &\geq 0 \quad \text{for } i = 1, \ldots, m, \\ h_j(x) &= 0 \quad \text{for } j = 1, \ldots, p, \\ \text{Complementary slackness:} \quad \lambda_i \cdot g_i(x) &= 0 \quad \text{for } i = 1, \ldots, m. \end{aligned}$$

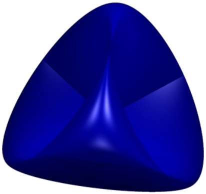
Notions of duality

Convex geometry



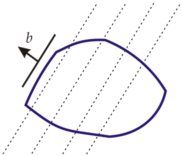
Dual body P^{Δ}

Algebraic geometry



Dual varieties X^*

Optimization



Dual problem: $\underset{y \in \mathcal{T}}{\text{Maximize}}\ b^T y$

Outline

1. Different notions of duality
2. Algebraic geometry and optimization
   - Relation
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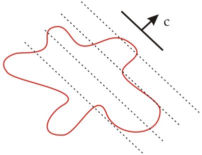
   - Example
3. Convex and algebraic geometry
   - Relation
   - Application
   - Example
4. Optimization and convex geometry
   - Relation
   - Application
   - Example

Algebraic geometry and optimization

Consider the following optimization problem:

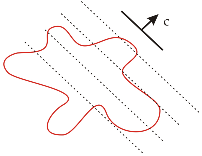


$$c_0 = \underset{x}{\text{Minimize}}\ c^T x$$
$$\text{s.t.}$$
$$x \in X = \{v \in \mathbb{R}^n \mid h_1(v) = \cdots = h_p(v) = 0\} \tag{3}$$

with compact, smooth and irreducible algebraic variety X and dual variety $X^* = V(\phi)$ (a hypersurface defined by $\phi(u_0, \ldots, u_n) = 0$).

Algebraic geometry and optimization

Consider the following optimization problem:



$$c_0 = \underset{x}{\text{Minimize}}\ c^T x \quad \text{s.t.} \quad x \in X = \{v \in \mathbb{R}^n \mid h_1(v) = \cdots = h_p(v) = 0\} \qquad (3)$$

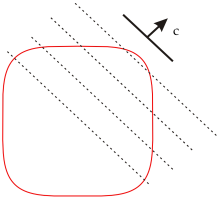
with compact, smooth and irreducible algebraic variety X and dual variety $X^* = V(\phi)$ (a hypersurface defined by $\phi(u_0, \ldots, u_n) = 0$).

Theorem [R., Sturmfels, 2010]:

The optimal value function for the optimization problem (3) is an algebraic function given by $\phi(-c_0, c_1, \ldots, c_n) = 0$.

Example: Optimization over the TV screen

$$c_0 = \text{Minimize}_x \; c_1 x_1 + c_2 x_2$$
$$\text{s.t.}$$
$$x \in X = V(x_1^4 + x_2^4 - 1).$$



Optimality condition:

$$\begin{aligned} c_1 &= \lambda_1 4 x_1^3 \\ c_2 &= \lambda_1 4 x_2^3 \\ 1 &= x_1^4 + x_2^4 \\ c_0 &= c_1 x_1 + c_2 x_2. \end{aligned}$$

Elimination of x, λ yields

$$\begin{aligned} \phi(-c_0, c_1, c_2) = & c_1^{12} + 3c_1^8 c_2^4 + 3c_1^4 c_2^8 + c_2^{12} - 3c_0^4 c_1^8 + 21 c_0^4 c_1^4 c_2^4 \\ & - 3c_0^4 c_2^8 + 3c_0^8 c_1^4 + 3c_0^8 c_1^4 - c_0^{12}. \end{aligned}$$

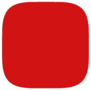
Convex and algebraic geometry

Def.: Algebraic boundary

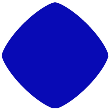
The algebraic boundary $\partial_a P$ of a convex body P is the *Zariski closure* ∂P.

Example: The 4-norm ball

$$P = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^4 + y^4 \leq 1\right\}$$



$$P^\Delta = \left\{(a, b) \in \mathbb{R}^2 \,\middle|\, |a|^{4/3} + |b|^{4/3} \leq 1\right\}$$



with

$$\partial_a P \;=\; V\left(x^4 + y^4 - 1\right)$$

and

$$\partial_a P^\Delta \;=\; V\left(a^{12} + 3a^8 b^4 + 3a^4 b^8 + b^{12} - 3a^8 + 21a^4 b^4 - 3b^8 + 3a^4 + 3b^4 - 1\right).$$

Application: Computing convex hulls

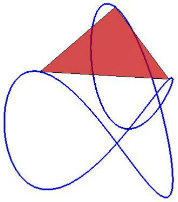
Theorem [Ranestad, Sturmfels, 2010]:

Let $P = \mathrm{conv} X$ the convex hull of a smooth algebraic variety X. Then

$$\partial_a P \subseteq \bigcup_k (X^{[k]})^*.$$

where $X^{[k]} = \overline{\{\text{hyperplanes tangent to } X \text{ at } k \text{ smooth points}\}}$.

Note: $X^{[1]} = X^*$ and $(X^{[1]})^* = (X^*)^* = X$.



A tritangent plane in $X^{[3]}$.

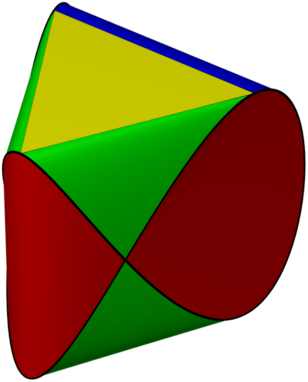
Example: Convex hull of a space curve

Algebraic curve $X = (\cos(\theta), \cos(2\theta), \sin(3\theta))$ in $\mathbb{R}^3$ or equivalently $X = V(h_1, h_2)$ with

$$h_1 = 2x^2 - y - 1,$$
$$h_2 = 4y^3 + 2z^2 - 3y - 1.$$

Convex hull:

- Two tritangent planes $z = 1$ and $z = -1$ (components of $(X^{[3]})^*$)
- Edge surface $(X^{[2]})^*$ with three irreducible components:
  - Quadratic surface $V(h_1)$
  - Cubic surface $V(h_2)$
  - Surface of degree 16



Curve X and its convex hull.

$$
\begin{aligned}
h_3 = {} & -419904x^{14}y^2 + 664848x^{12}y^4 - 419904x^{10}y^6 + 132192x^8y^8 - 20736x^6y^{10} + 1296x^4y^{12} \\
& - 46656x^{14}z^2 + 373248x^{12}y^2z^2 - 69984x^{10}y^4z^2 - 22464x^8y^6z^2 + 4320x^6y^8z^2 + 31104x^{12}z^4 \\
& + 5184x^{10}y^2z^4 + 4752x^8y^4z^4 + 1728x^{10}z^6 + 699840x^{14}y - 46656x^{12}y^3 - 902016x^{10}y^5 \\
& + 694656x^8y^7 - 209088x^6y^9 - 1150848x^{10}y^3z^2 + 279936x^8y^5z^2 + 17280x^6y^7z^2 - 4032x^4y^9z^2 \\
& - 98496x^{10}yz^4 + 27072x^4y^{11} - 1152x^2y^{13} - 419904x^{12}yz^2 - 25920x^8y^3z^4 - 4608x^6y^5z^4 \\
& - 1728x^8yz^6 - 291600x^{14} - 169128x^{12}y^2 - 256608x^{10}y^4 + 956880x^8y^6 - 618192x^6y^8 \\
& + 148824x^4y^{10} - 13120x^2y^{12} + 256y^{14} + 392688x^{12}z^2 + 671976x^{10}y^2z^2 + 1454976x^8y^4z^2 \\
& - 292608x^6y^6z^2 - 4272x^4y^8z^2 + 1016x^2y^{10}z^2 - 116208x^{10}z^4 + 135432x^8y^2z^4 + 18144x^6y^4z^4 \\
& + 1264x^4y^6z^4 - 5616x^8z^6 + 504x^6y^2z^6 - 1108080x^{12}y + 925344x^{10}y^3 + 215136x^8y^5 \\
& - 672192x^6y^7 + 331920x^4y^9 - 54240x^2y^{11} + 2304y^{13} + 273456x^{10}yz^2 + 282528x^8y^3z^2 \\
& - 1185408x^6y^5z^2 + 149376x^4y^7z^2 - 368x^2y^9z^2 - 32y^{11}z^2 + 273456x^8yz^4 - 67104x^6y^3z^4 \\
& - 4704x^4y^5z^4 - 64x^2y^7z^4 + 4752x^6yz^6 - 32x^4y^3z^6 + 747225x^{12} + 636660x^{10}y^2 \\
& - 908010x^8y^4 - 65340x^6y^6 + 291465x^4y^8 - 101712x^2y^{10} + 8256y^{12} - 818100x^{10}z^2 \\
& - 1405836x^8y^2z^2 - 905634x^6y^4z^2 + 583824x^4y^6z^2 - 39318x^2y^8z^2 + 368y^{10}z^2 + 193806x^8z^4 \\
& - 282996x^6y^2z^4 + 15450x^4y^4z^4 + 716x^2y^6z^4 + y^8z^4 + 6876x^6z^6 - 1140x^4y^2z^6 + 2x^2y^4z^6 \\
& + x^4z^8 + 507384x^{10}y - 809568x^8y^3 + 569592x^6y^5 - 27216x^4y^7 - 71648x^2y^9 + 13952y^{11} \\
& \ldots + 98 \text{ other terms} \ldots
\end{aligned}
$$

Optimization and convex geometry

SDP: Linear optimization over a spectrahedron[1]

$$p^* = \underset{x}{\text{Minimize}}\ c^T x$$
$$\text{s.t.}$$
$$x \in P = \{v \in \mathbb{R}^m \mid Q(v) = Q_0 + \sum_i v_i Q_i \succeq 0\}$$

Proposition:

The (Lagrange) dual optimization problem can be written as:

$$d^* = \underset{c_0}{\text{Maximize}}\ c_0$$
$$\text{s.t.}$$
$$\frac{1}{c_0} c \in P^\Delta.$$

[1] Assumption: $Q_0 \succ 0$.

Primal SDP:

$$p^* := \underset{X \in \mathcal{S}^n_+}{\text{Minimize}} \langle B, Q_0 - X \rangle \text{ subject to } X \in (Q_0 + \mathcal{W}) \cap \mathcal{S}^n_+$$

Dual:

$$d^* := \underset{Y \in \mathcal{S}^n_+}{\text{Maximize}} \langle Q_0, Y \rangle \text{ subject to } Y \in (B + \mathcal{W}^\perp) \cap \mathcal{S}^n_+$$

with $B, Q_0 \succ 0$, $\mathcal{W} = \text{span}(Q_1, \ldots, Q_n)$ and $\langle Q_i, B \rangle = c_i$ for $i = 1, \ldots, m$.

Optimality condition: Assuming strict feasibility of primal and dual optimization problem, an optimal pair of solutions (X, Y) satisfy the following KKT conditions:

$$X \in (Q_0 + \mathcal{W}) \cap \mathcal{S}^n_+$$
$$Y \in (B + \mathcal{W}^\perp) \cap \mathcal{S}^n_+$$
$$X \cdot Y = 0 \quad \text{(complementary slackness).}$$

Optimality condition: Assuming strict feasibility of primal and dual optimization problem, an optimal pair of solutions (X, Y) satisfy the following KKT conditions:

$$X \in (Q_0 + \mathcal{W}) \cap \mathcal{S}^n_+ \qquad \underset{\rightarrow}{homogenize}: \; X \in \mathcal{U} = \mathbb{R}Q_0 + \mathcal{W}$$

$$Y \in (B + \mathcal{W}^\perp) \cap \mathcal{S}^n_+ \qquad Y \in \mathcal{L}^\perp = \mathbb{R}B + \mathcal{W}^\perp$$

$$X \cdot Y = 0 \quad \text{(complementary slackness).}$$

Optimality condition: Assuming strict feasibility of primal and dual optimization problem, an optimal pair of solutions (X, Y) satisfy the following KKT conditions:

$$
\begin{aligned}
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Y &\in (B + \mathcal{W}^\perp) \cap \mathcal{S}^n_+ \\
X \cdot Y &= 0 \quad \text{(complementary slackness).}
\end{aligned}
\qquad \underset{\rightarrow}{\textit{homogenize}}: \quad
\begin{aligned}
X &\in \mathcal{U} = \mathbb{R}Q_0 + \mathcal{W} \\
Y &\in \mathcal{L}^\perp = \mathbb{R}B + \mathcal{W}^\perp
\end{aligned}
$$

"Algebraic" SDP:

Given any a flag of linear subspaces $\mathcal{U} \subset \mathcal{L} \subset \mathcal{S}^n$ with $\dim(\mathcal{U}/\mathcal{L}) = 2$, find the unique semidefinite point $(X, Y) \in \mathcal{U} \times \mathcal{L}^\perp$ in $X \cdot Y = 0$.

Nie, Ranestad and Sturmfels showed the following decomposition into irreducible components:

$$\{X \cdot Y = 0\} \; = \; \bigcup_{r=1}^{n-1} \{X \cdot Y = 0\}^r \quad \subset \; \mathbb{P}(\mathcal{S}^n) \times \mathbb{P}(\mathcal{S}^n).$$

where $\{X \cdot Y = 0\}^r$ denotes the subvariety of pairs (X, Y) with $\mathrm{rank}(X) \leq r$ and $\mathrm{rank}(Y) \leq n - r$.

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Theorem [R., Sturmfels, 2010]:

For a (sufficiently generic) spectrahedron P the algebraic boundary of its dual body P^Δ is given by

$$\partial_a P^\Delta \subseteq \bigcup_{r \in \text{(Pataki range)}} \{X \in \mathcal{L} \mid \mathrm{rank}(X) \leq r\}^*$$

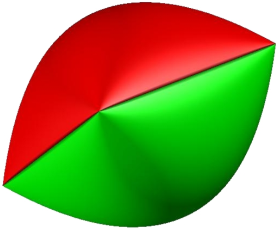
Example: Honest pillow – Dual body

Honest pillow:

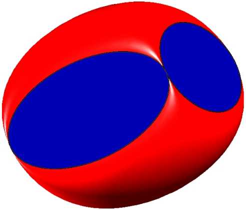
$$P = \{ (x,y,z) \in \mathbb{R}^3 \mid Q(x,y,z) \succeq 0 \} \text{ with } Q(x,y,z) = \begin{pmatrix} 1 & x & 0 & x \\ x & 1 & y & 0 \\ 0 & y & 1 & z \\ x & 0 & z & 1 \end{pmatrix}.$$

Dual pillow:

$$P^{\Delta} = \{ (a,b,c) \in \mathbb{R}^3 \mid ax + by + cz \leq 1 \text{ for all } (x,y,z) \in P \}.$$



Honest pillow



Dual pillow

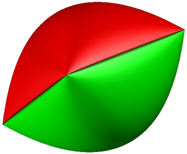
Example: Honest pillow – Faces

Pillow:

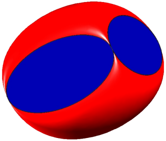
- Four 1-dimensional faces,
- Four singular 0-dimensional faces
- Two smooth families of 0-dimensional faces

Dual pillow:

- Four (isolated) faces of dimension 0
- Four 2 dimensional "ovals"
- Two smooth families of 0-dimensional faces



Honest pillow



Dual body

Example: Honest pillow – Algebraic boundary

Two components of $\partial_a P^\Delta$:

- Dual to the smooth patches of ∂P (with $\mathrm{rank}(Q(v)) = 3$):

  $$(c_2^2 + 2c_2c_3 + c_3^2) \cdot c_0^2 - c_1^2c_2^2 - c_1^2c_3^2 - c_2^4 - 2c_2^2c_3^2 - 2c_2c_3^3 - c_3^4 - 2c_2^3c_3 = 0.$$

- Dual to the singular points on ∂P (with $\mathrm{rank}(Q(v)) = 2$):

$$4(c_0 + \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 - \frac{\sqrt{2}}{2}c_3) \cdot (c_0 - \frac{\sqrt{2}}{2}c_1 - \frac{\sqrt{2}}{2}c_2 + \frac{\sqrt{2}}{2}c_3)$$
$$\cdot(c_0 - \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 - \frac{\sqrt{2}}{2}c_3) \cdot (c_0 + \frac{\sqrt{2}}{2}c_1 - \frac{\sqrt{2}}{2}c_2 + \frac{\sqrt{2}}{2}c_3) = 0.$$

Example: Honest pillow – Optimization

(Linear) Optimization over the honest pillow:

$$p^*(c_1, c_2, c_3) \;=\; \underset{(x,y,z)\in\mathbb{R}^3}{\text{Minimize}} \quad c_1x + c_2y + c_3z$$
$$\text{subject to} \quad Q(x,y,z) \succeq 0.$$

with associated dual problem:

$$d^*(c_1, c_2, c_3) \;=\; \underset{c_0\in\mathbb{R}}{\text{Maximize}} \quad c_0$$
$$\text{subject to} \quad \frac{1}{c_0} \cdot c \;\in\; P^{\Delta}.$$

Example: Honest pillow – Optimization

Optimal solution...

- ...is on the smooth part of P, if:

$$(c_2^2+2c_2c_3+c_3^2)\cdot c_0^2-c_1^2c_2^2-c_1^2c_3^2-c_2^4-2c_2^2c_3^2-2c_2c_3^3-c_3^4-2c_2^3c_3=0.$$

- ...is on a "corner" of P, if:

$$4\left(c_0+\frac{\sqrt{2}}{2}c_1+\frac{\sqrt{2}}{2}c_2-\frac{\sqrt{2}}{2}c_3\right)\cdot\left(c_0-\frac{\sqrt{2}}{2}c_1-\frac{\sqrt{2}}{2}c_2+\frac{\sqrt{2}}{2}c_3\right)$$
$$\cdot\left(c_0-\frac{\sqrt{2}}{2}c_1+\frac{\sqrt{2}}{2}c_2-\frac{\sqrt{2}}{2}c_3\right)\cdot\left(c_0+\frac{\sqrt{2}}{2}c_1-\frac{\sqrt{2}}{2}c_2+\frac{\sqrt{2}}{2}c_3\right)=0.$$

Questions?

Thank you very much for your attention!

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