On the projective dimension of edge ideals of chordal graphs

Kyouko Kimura (Osaka University/JST CREST)

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$G = (V(G), E(G))$: a finite graph
with no loop, no multiple edge.

$V = V(G)$: the vertex set of G.

$E(G)$: the edge set of G.

- G is *chordal* if each cycle of length > 3 has a chord.
- G is a *forest* if G contains no cycle.

$S = K[x : x \in V]$: a polynomial ring / a field K.
$\deg x = 1$.

The edge ideal of G:

$$I(G) = (x_i x_j \ : \ \{x_i, x_j\} \in E(G)).$$

A minimal graded free resolution of $S/I(G)$:

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{p,j}} \xrightarrow{d_p} \cdots \longrightarrow \bigoplus_j S(-j)^{\beta_{1,j}} \longrightarrow S \longrightarrow S/I(G) \longrightarrow 0.$$

$S = \bigoplus_n S_n$, $\quad$ S_n: nth homogeneous component of S.
$[S(-j)]_n = S_{n-j}$.

$\beta_{i,j} = \beta_{i,j}(S/I(G))$: (i,j)-th graded betti number of $S/I(G)$.
$p = \mathrm{pd}_S\, S/I(G)$: the projective dimension of $S/I(G)$.
$\mathrm{reg}\, S/I(G) = \max\{j - i : \beta_{i,j} \neq 0\}$: the regularity of $S/I(G)$.

Problem 1. *Describe these invariants in terms of combinatorial data of G.*

Known:

- reg $S/I(G)$ for chordal graphs (Zheng, Hà and Van Tuyl).
- $\mathrm{pd}_S\, S/I(G)$ for forests (Zheng).

Results:

- $\mathrm{pd}_S\, S/I(G)$ for chordal graphs.
- $\beta_{i,j}(S/I(G))$ for forests.

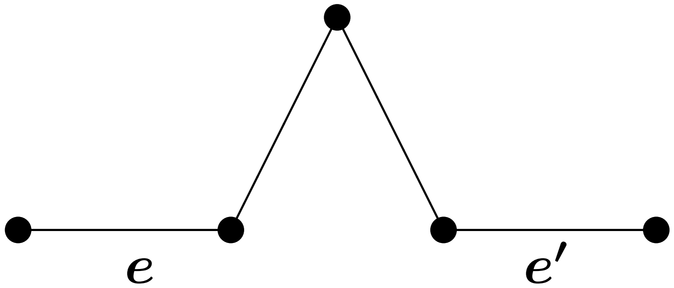
★ **Notation & definition.**

Definition 2 (Hà and Van Tuyl). **Let e, e' be two distinct edges of G. Suppose that e, e' belong to the same connected component of G. Then the *distance* of e, e' (in G) is defined by**

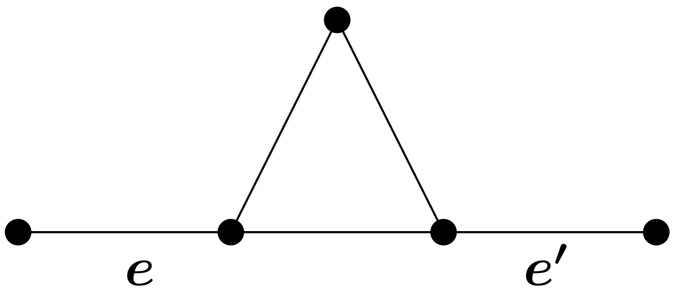
$$\mathrm{dist}_G(e, e') := \min \left\{ \ell \ : \ \begin{array}{l} \exists e_0 = e, e_1, \ldots, e_\ell = e' \\ \text{s.t. } e_{i-1} \cap e_i \neq \emptyset, \ e_i \in E(G) \end{array} \right\}.$$

When e, e' belong to the different connected component of G, then we set $\mathrm{dist}_G(e, e') = \infty$.

We say that e and e' are *3-disjoint* in G if $\mathrm{dist}_G(e, e') \geq 3$.

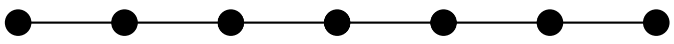


$$\mathrm{dist}_G(e, e') = 3$$

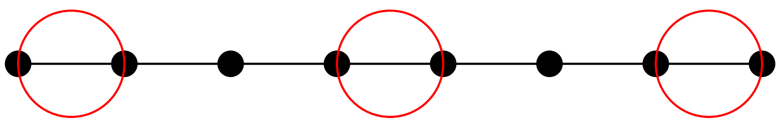


$$\mathrm{dist}_G(e, e') = 2$$

Theorem 3 (Zheng, Hà and Van Tuyl). ***Let G be a chordal graph. Then the regularity of $S/I(G)$ coincides with the maximum number of pairwise 3-disjoint edges of G.***



$\mathrm{reg}\, S/I(G) = 2$

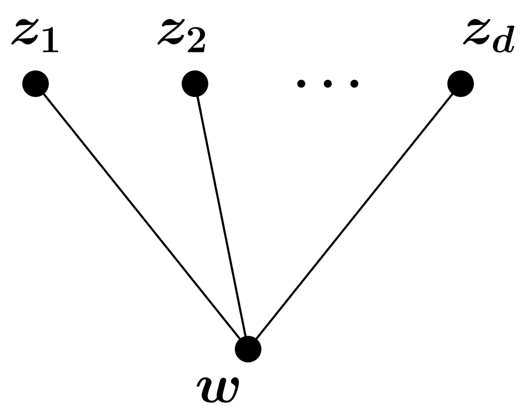


$\mathrm{reg}\, S/I(G) = 3$

Definition 4 (Zheng). **The graph B with $V(B) = \{w, z_1, \ldots, z_d\}$ and $E(B) = \{\{w, z_1\}, \ldots, \{w, z_d\}\}$ $(d \geq 1)$ called a *bouquet*.**

On the above bouquet B, the vertex w is called a *root* of B and the vertices z_i *flowers* of B, edges $\{w, z_i\}$ *stems* of B.

B:

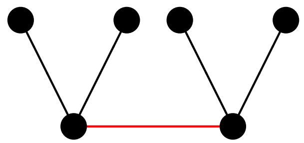


Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a set of bouquets those are subgraphs of G

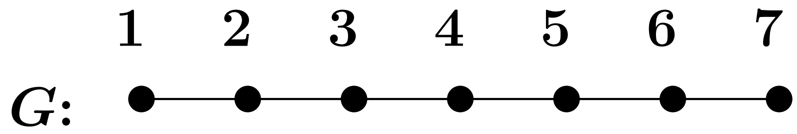
Definition 5.

(1) We say $\mathcal{B}$ is *strongly disjoint* in G if for any $i \neq j$, bouquets B_i, B_j contain no common vertex, and there exists the set of edges $\{s_1, \ldots, s_k\}$ where s_i is a stem of B_i and s_i, s_j are 3-disjoint in G for all $i \neq j$.

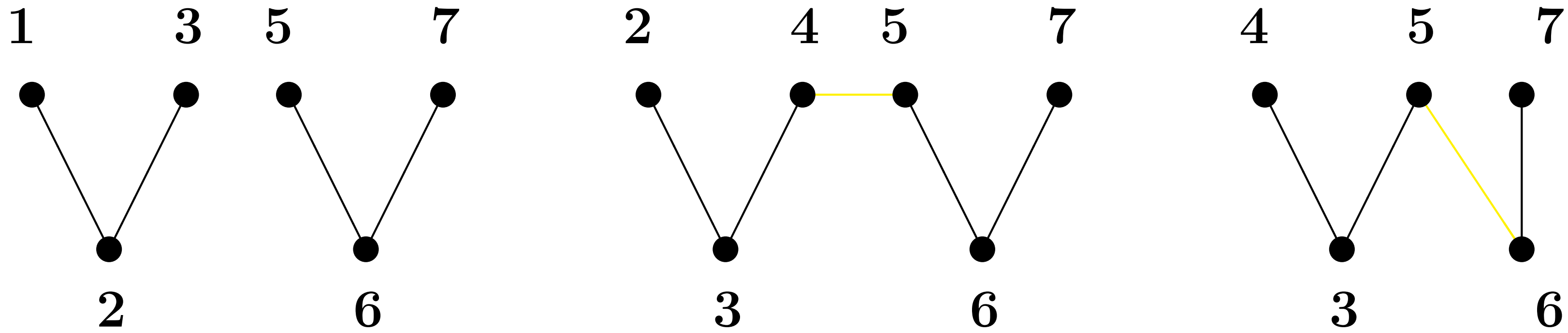
(2) We say $\mathcal{B}$ is *semi-strongly disjoint* in G if for any $i \neq j$, bouquets B_i, B_j contain no common vertex and the roots of B_i, B_j have no common edge in G.



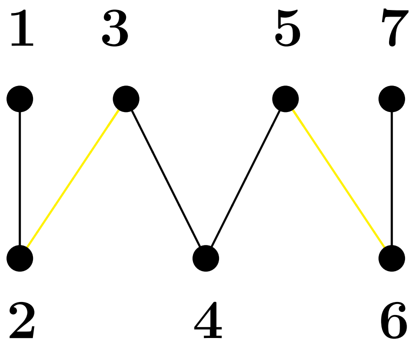
not semi-strongly disjoint



Strongly disjoint set of bouquets:



Not-strongly, semi-strongly disjoint set of bouquets:



★ **First Result.**

For a bouquet B, we denote by $n(B)$, the number of flowers of B. For a set of bouquets $\mathcal{B} = \{B_1, \dots, B_k\}$, we set $n(\mathcal{B}) = n(B_1) + \cdots + n(B_k)$.
Set

$$d = \max \left\{ n(\mathcal{B}) \ : \ \begin{array}{l} \mathcal{B} \text{ is a semi-strongly disjoint set} \\ \text{of bouquets of } G \end{array} \right\}.$$

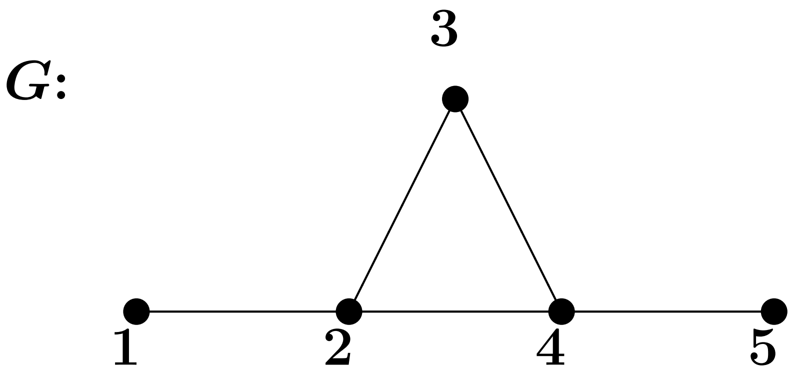
$$d' = \max\{n(\mathcal{B}) \ : \ \mathcal{B} \text{ is a strongly disjoint set of bouquets of } G\}.$$

Then $d' \leq d$.

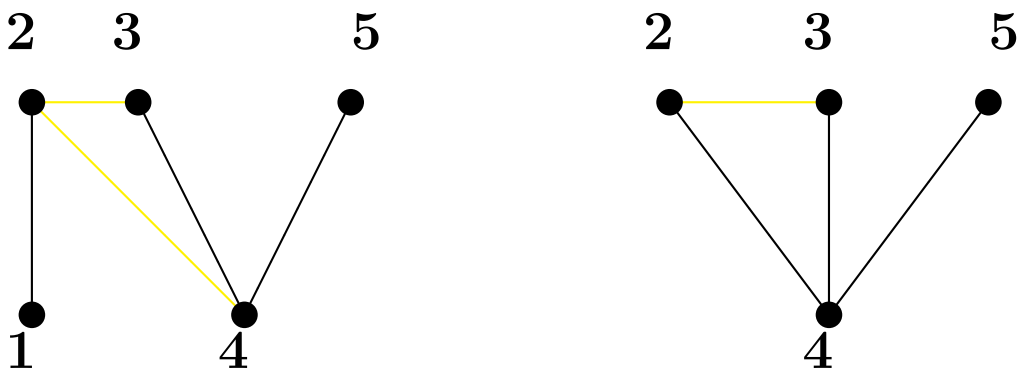
Theorem 6. *Let G be a chordal graph. Then*

$$\operatorname{pd}_S S/I(G) = d = d'.$$

Remark 7. Zheng proved this theorem when G was a forest.



$$\mathbf{pd}_S\, S/I(G) = 3.$$



★ **Second Result.**

Let $\mathcal{B}$ be a set of bouquets. We denote by

$R(\mathcal{B})$**, the set of roots of the bouquets in** $\mathcal{B}$**,**
$F(\mathcal{B})$**, union of the sets of flowers of the bouquets in** $\mathcal{B}$**.**

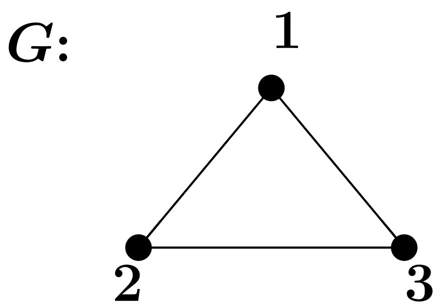
Definition 8. **We say that a graph G contains strongly disjoint set of bouquets of type (i, j) if there exists a strongly disjoint set of bouquets $\mathcal{B}$ in G such that**

$$R(\mathcal{B}) \cup F(\mathcal{B}) = V(G);$$
$$\#F(\mathcal{B}) = i; \qquad \#R(\mathcal{B}) = j.$$

Theorem 9. *Let G be a forest. Then $\beta_{i,i+j}(S/I(G))$ coincides with the number of subsets W of $V = V(G)$ such that G_W contains a strongly disjoint set of bouquets of type (i, j).*

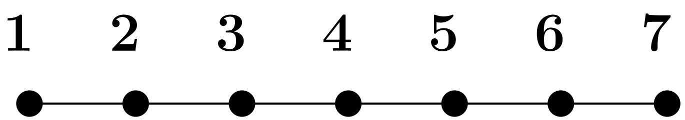
Remark 10. **If G is chordal, then the claim of Theorem 9 is false.**

For example,



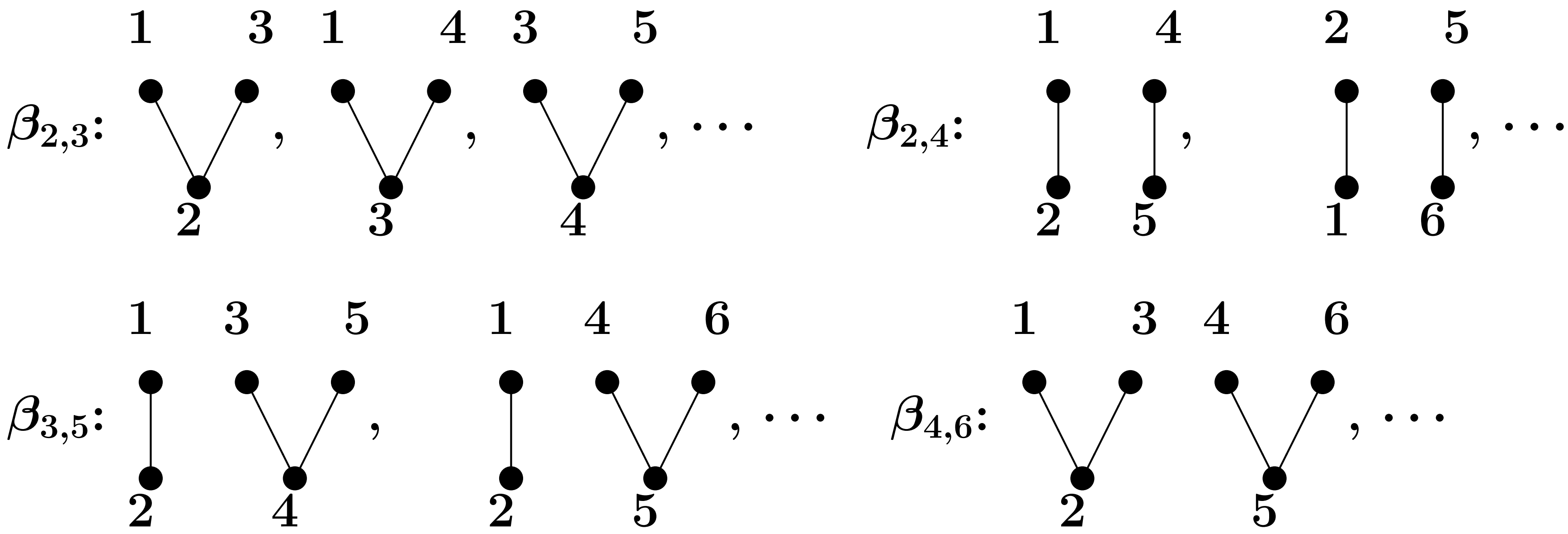
Then $\beta_{2,2+1}(S/I(G)) = 2$. But a subset W of $V(G) = \{1, 2, 3\}$ with $\#W = 2 + 1 = 3$ is only $V(G)$.

G:



$\beta_{i,i+j}(S/I(G))$:

$j \setminus i$	0	1	2	3	4
0	1				
1		6	5		
2			6	9	3



★ **Key Lemma.**

Lemma 11 (Hà and Van Tuyl). *Let G be a chordal graph. Suppose that $e = \{u, v\}$ is an edge of G such that $G_{N(v)}$ is a complete graph. Let $t = \#N(u) - 1$ and G' the subgraph of G with*

$$E(G') = \{e' \in E(G) \ : \ \mathrm{dist}_G(e, e') \geq 3\}.$$

Then both of $G \setminus e$ and G' are chordal and

$$\beta_{i,i+j}(S/I(G)) = \beta_{i,i+j}(S/I(G \setminus e)) + \sum_{\ell=0}^{i-1} \binom{t}{\ell} \beta_{i-1-\ell,(i-1-\ell)+(j-1)}(S/I(G')),$$

$$\beta_i(S/I(G)) = \beta_i(S/I(G \setminus e)) + \sum_{\ell=0}^{i-1} \binom{t}{\ell} \beta_{i-1-\ell}(S/I(G')).$$