Hyperdeterminantal Total Positivity

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Total positivity

$K : \mathbb{R}^2 \to \mathbb{R}$ is totally positive of order d if, for all $n = 1, \ldots, d$,

$$\det\big(K(u_i, v_j)\big)_{n\times n} := \begin{vmatrix} K(u_1, v_1) & \cdots & K(u_1, v_n) \\ \vdots & & \vdots \\ K(u_n, v_1) & \cdots & K(u_n, v_n) \end{vmatrix} \geq 0$$

whenever $u_1 > \cdots > u_d$ and $v_1 > \cdots > v_d$

The basic idea underlying classical TP:

Square matrices A for which all minors are nonnegative

Fundamental examples of TP kernels

$$K(u,v) = \exp(uv), \quad u, v \in \mathbb{R}$$

Random walks, hypergeometric functions of matrix argument, statistical inference, Lie groups, random matrices, ...

$$K(u,v) = \begin{cases} 1, & \text{if } u \geq v \\ 0, & \text{otherwise} \end{cases}$$

Approximation theory, game theory, economics, probability inequalities, combinatorics, ...

Karlin, “Total Positivity,” 1968

Schoenberg, Gantmacher, Krein, Pólya, Szegö, Karlin, McGregor, Whitney, Aissen, Hirschman, Edrei, Motzkin, Studden, Ando, Cryer, Loewner, Pinkus, Rinott, Lusztig, Fomin, Brenti, Williams, Gross, Richards, . . .

Statistics, mathematics, game theory, economics, physics, computer science

Generalizations of total positivity for kernels on $\mathbb{R}^p$

Karlin and Rinott (1980): Multivariate TP_2

Rinott and Saks (1993)

Correlation inequalities for random vectors

FKG inequality

Gross and R. (1995): TP and finite reflection groups

Hypergeometric functions of matrix argument

R. (2004): Generalizations of the FKG inequality

$\mathfrak{S}_n$: The symmetric group on n symbols

Fundamental Weyl chamber:

$$\mathcal{C}_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_1 > \cdots > t_n\}$$

TP$_d$: For $(u_1, \dots, u_d)$ and $(v_1, \dots, v_d) \in \mathcal{C}_d$, all minors of the $d \times d$ matrix $\big(K(u_i, v_j)\big)$ are nonnegative:

$$\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} K(u_j, v_{\sigma \cdot j}) \geq 0$$

Gross and R. (1995): Replace $\mathfrak{S}_n$ by W, a finite reflection group; replace $\mathcal{C}_n$ by the corresponding fundamental Weyl chamber

μ: A nice (positive, Borel) measure on $\mathbb{R}$

Basic Composition Formula: If K, L are TP$_d$ then so is the kernel,

$$M(u,v) = \int_{\mathbb{R}} K(u,t)L(t,v)\,\mathrm{d}\mu(t)$$

Binet-Cauchy formula for determinants:

$$\det\big(M(u_i,v_j)\big) = \int_{\mathcal{C}} \det\big(K(u_i,t_j)\big)\ \det\big(L(t_i,v_j)\big) \prod_{j=1}^{n} \mathrm{d}\mu(t_j)$$

Hyperdeterminants

$i_1, \ldots, i_{2m}$: Indices in $\{1, \ldots, n\}$

$A(i_1, \ldots, i_{2m}) \in \mathbb{C}$ for each $(i_1, \ldots, i_{2m})$

Multidimensional array: $A = \big(A(i_1, \ldots, i_{2m})\big)_{n\times\cdots\times n}$

Cayley (1843,1845,1846): The hyperdeterminant of A

$$\mathrm{Det}\big(A(i_1, \ldots, i_{2m})\big)$$

$$:= \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_{2m}\in\mathfrak{S}_n} \prod_{k=1}^{2m} \operatorname{sgn}(\sigma_k) \cdot \prod_{j=1}^{n} A(\sigma_1 \cdot j, \ldots, \sigma_{2m} \cdot j)$$

Sokolov, Lecat, Gasparyan, Oldenburger, Rice, Hval, ...

E. Pascal, “Die Determinanten,” 1900

Gel’fand, Kapranov, Zelevinsky, “Discriminants, Resultants, and Multidimensional Determinants,” 1994

Connections with Gröbner bases

Matsumoto, Evans, Gottlieb, Sturmfels

$m = 1$: Hyperdeterminant is the classical determinant

Many properties of determinants extend to hyperdeterminants

Laplace expansion

Fix $0 \le r \le n$, $I_1 = (l_{1,1}, \dots, l_{1,r})$, $1 \le l_{1,1} \le \cdots \le l_{1,r} \le n$

Define $\bar{I}_1 = \{1, \dots, n\} \setminus I_1$; then,

$$\mathrm{Det}(A) = \sum_{I_2,\dots,I_{2m}} \prod_{k=2}^{2m} \mathrm{sgn}(\sigma_k) \cdot \mathrm{Det}\left(A \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_{2m} \end{pmatrix} \right) \mathrm{Det}\left(A \begin{pmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \vdots \\ \bar{I}_{2m} \end{pmatrix} \right)$$

$I_2 = (l_{2,1}, \dots, l_{2,r}), \dots, I_{2m} = (l_{2m,1}, \dots, l_{2m,r}) \in \{1, \dots, n\}^r$

σ_k is the permutation restoring $\{I_i, \bar{I}_i\}$ to standard order

The hyperdeterminant is a multi-sum of classical determinants

$$\mathrm{Det}\big(A(i_1,\ldots,i_{2m})\big) = \sum_{\sigma_1,\ldots,\sigma_{2m-2}\in\mathfrak{S}_2} \prod_{k=1}^{2m-2} \operatorname{sgn}(\sigma_k) \cdot \det\big(A(\sigma_1\cdot i,\ldots,\sigma_{2m-2}\cdot i, i, j)\big)_{n\times n}$$

The case $d = m = 2$:

$$\text{Det}\big(A(i_1,\ldots,i_4)\big)_{2\times\cdots\times 2} = \begin{vmatrix} A(1,1,1,1) & A(1,1,1,2) \\ A(2,2,2,1) & A(2,2,2,2) \end{vmatrix}$$

$$- \begin{vmatrix} A(1,2,1,1) & A(1,2,1,2) \\ A(2,1,2,1) & A(2,1,2,2) \end{vmatrix}$$

$$- \begin{vmatrix} A(2,1,1,1) & A(2,1,1,2) \\ A(1,2,2,1) & A(1,2,2,2) \end{vmatrix}$$

$$+ \begin{vmatrix} A(2,2,1,1) & A(2,2,1,2) \\ A(1,1,2,1) & A(1,1,2,2) \end{vmatrix}$$

Hyperdeterminantal total positivity

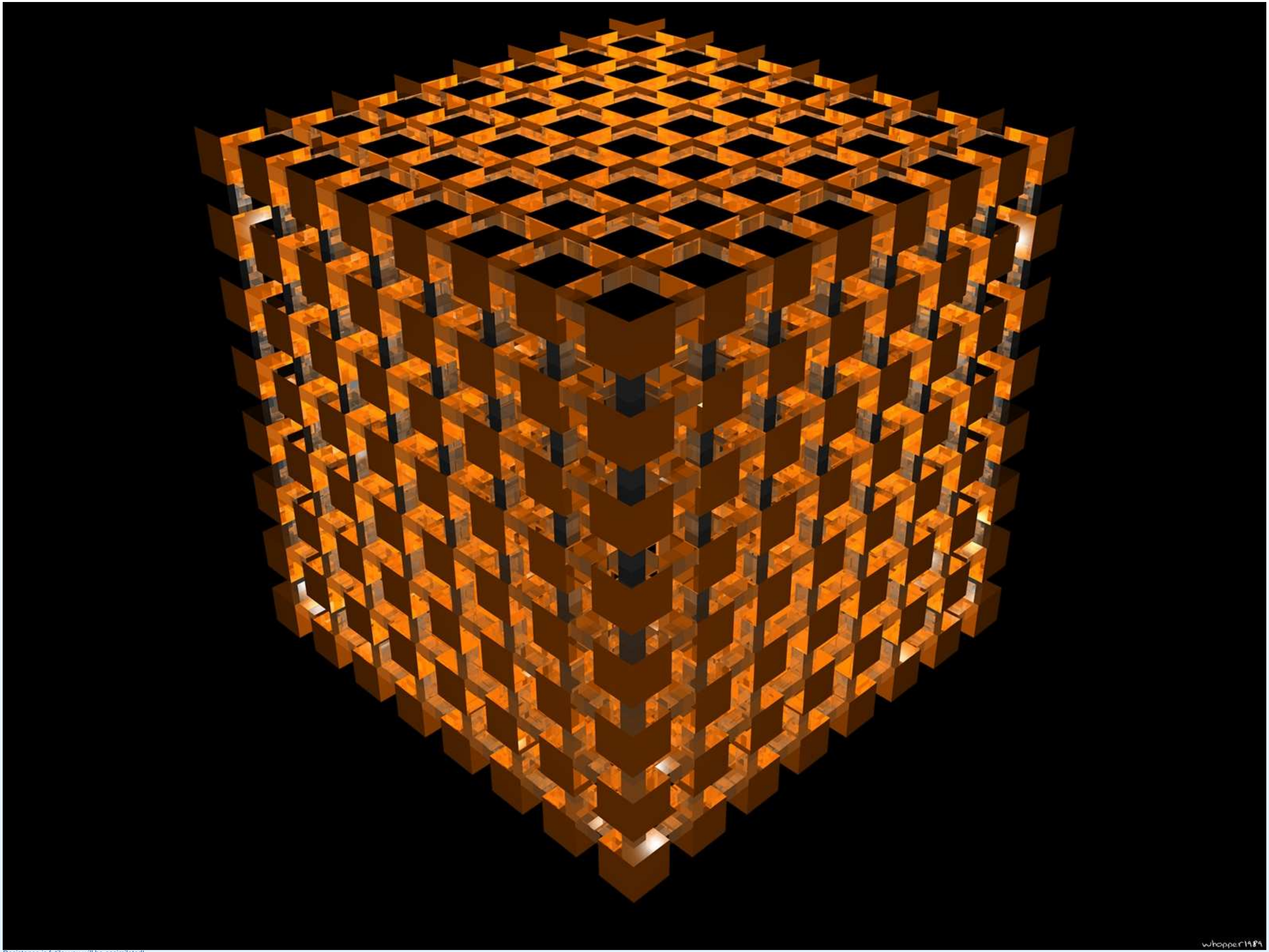
Definition: $K : \mathbb{R}^{2m} \to \mathbb{R}_+$ is HTP$_d$ if, for all $n = 1, \ldots, d$,

$$\text{Det}\big(K(x_{1,i_1}, x_{2,i_2}, \ldots, x_{2m,i_{2m}})\big)_{1 \le i_1, \ldots, i_{2m} \le n} \ \ge \ 0$$

for all vectors $(x_{k,1}, \ldots, x_{k,n}) \in \mathcal{C}_n,\ 1 \le k \le 2m$

The basic idea: An array is HTP if its sub-arrays all have nonnegative hyperdeterminant

If all $\text{Det}(K) > 0$ then K is called strictly HTP$_d$ (SHTP$_\infty$)



Resistance is futile; you will be assimilated!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Recall that $K(u,v) = \exp(uv)$ is STP$_\infty$ on $\mathbb{R}^2$

Generalization to multidimensional arrays

Theorem: $K(t_1,\ldots,t_{2m}) = \exp(t_1 \cdots t_{2m})$ is SHTP$_\infty$ on $\mathbb{R}^{2m}$:

$$\text{Det}\big(K(x_{1,i_1}, x_{2,i_2}, \ldots, x_{2m,i_{2m}})\big)_{n\times\cdots\times n} \; > \; 0$$

for all $n \geq 1$ and all vectors $(x_{k,1},\ldots,x_{k,n}) \in \mathcal{C}_n, \; 1 \leq k \leq 2m$

Extensions to generalized hypergeometric series

$$K(t_1,\ldots,t_{2m}) = {}_pF_q\binom{a_1,\ldots,a_p}{b_1,\ldots,b_q}; t_1\cdots t_{2m}\Big)$$

Functions $\phi_{k,i} : \mathbb{R} \to \mathbb{C}$, $1 \le k \le 2m$, $1 \le i \le n$

$$A(i_1, \ldots, i_{2m}) := \int_{\mathbb{R}} \phi_{1,i_1}(x)\phi_{2,i_2}(x) \cdots \phi_{2m,i_{2m}}(x) \; \mathrm{d}\mu(x)$$

The Binet-Cauchy formula for hyperdeterminants:

$$\mathrm{Det}\big(A(i_1, \ldots, i_{2m})\big)$$
$$= \int_{\mathcal{C}_n} \prod_{k=1}^{2m} \det\big(\phi_{k,i}(x_j)\big)_{1\le i,j\le n} \cdot \prod_{j=1}^{n} \mathrm{d}\mu(x_j)$$

Basic Composition Formula for HTP kernels

Kernels: $L_k(i,t) = \phi_{k,i}(t)$, $1 \le k \le 2m$

Construct the $n \times \cdots \times n$ array,

$$A(i_1, \ldots, i_{2m}) := \int_{\mathbb{R}} \phi_{1,i_1}(x)\phi_{2,i_2}(x) \cdots \phi_{2m,i_{2m}}(x) \; \mathrm{d}\mu(x)$$

If the kernels L_k all are TP$_d$ then $\big(A(i_1, \ldots, i_{2m})\big)$ is HTP$_d$

TP_∞ kernel: $K(u,v) = \begin{cases} 1, & \text{if } u \geq v \\ 0, & \text{otherwise} \end{cases}$

$$\det\left(K(u_i, v_j)\right)_{n\times n}$$
$$= \begin{cases} 1, & \text{if } u_1 \geq v_1 > u_2 \geq v_2 > \cdots > u_n \geq v_n \\ 0, & \text{otherwise} \end{cases}$$

Generalization to HTP

The kernel

$$K(t_1,\ldots,t_{2m}) = \begin{cases} 1, & \text{if } t_1 \geq \cdots \geq t_{2m} \\ 0, & \text{otherwise} \end{cases}$$

is HTP$_\infty$. Moreover, for $x_k = (x_{k,1},\ldots,x_{k,n}) \in \mathcal{C}_n,\ 1 \leq k \leq 2m$,

$$\text{Det}\big(K(x_{1,i_1},\ldots,x_{2m,i_{2m}})\big)_{n\times\cdots\times n}$$
$$= \begin{cases} 1, & \text{if } x_{1,1} \geq \cdots \geq x_{2m,1} > x_{1,2} \geq \cdots \geq x_{2m,2} > \cdots \\ 0, & \text{otherwise} \end{cases}$$

Applications to statistics

Probability inequalities

Spectral properties

Resistance is futile; you will be assimilated!