Roots of Ehrhart polynomials arising from graphs and posets

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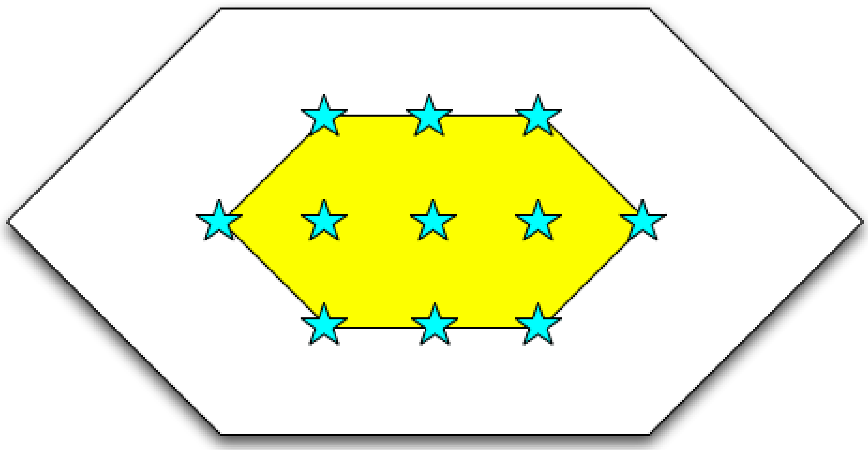
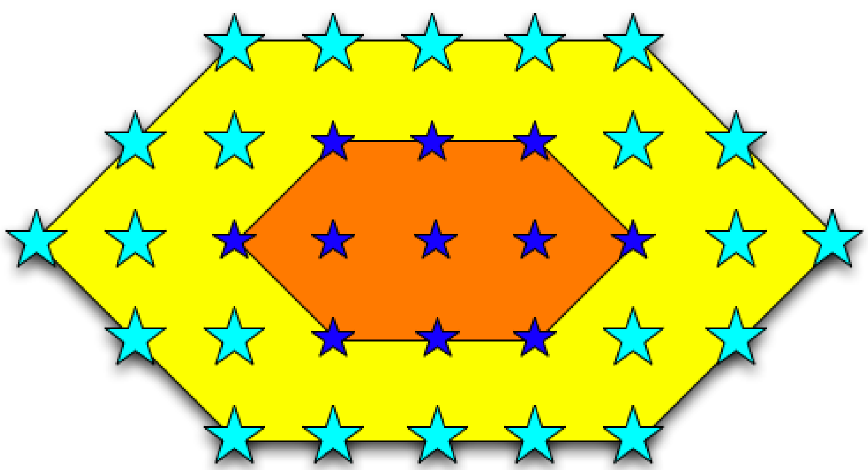
What is Ehrhart Polynomial?



The polynomial counts the number of integer points in multiplied polytopes.

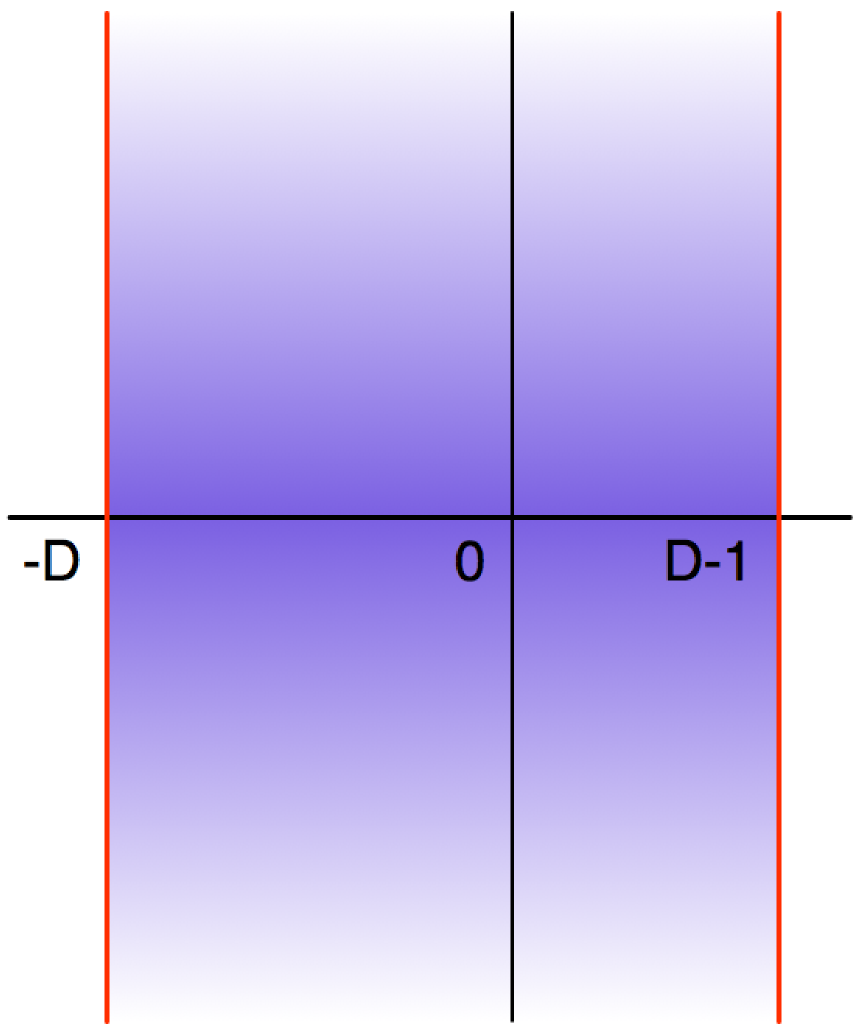
What is Ehrhart Polynomial?



The polynomial counts the number of integer points in multiplied polytopes.

i(1)=11, i(2)=33, etc.

The Conjecture of Beck *et al.*



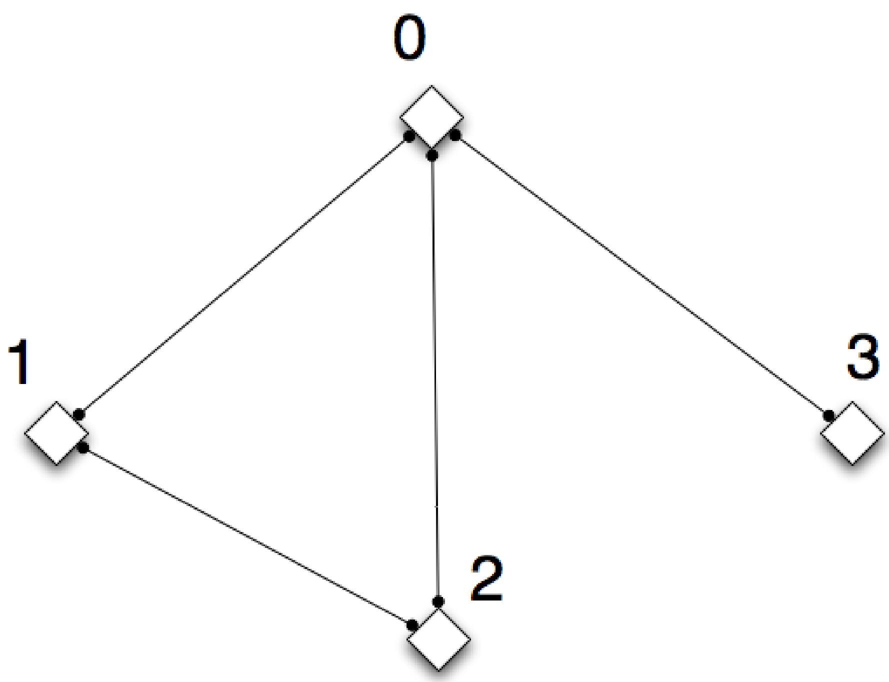
Any root α of Ehrhart polynomials of polytopes with dimension *D* have their real part in the range:

$$-D \leqq \mathrm{Re}(\alpha) \leqq D-1.$$

Polytopes arising from Graphs and Posets

- Edge Polytopes
- Symmetric Edge Polytopes
- Order Polytopes

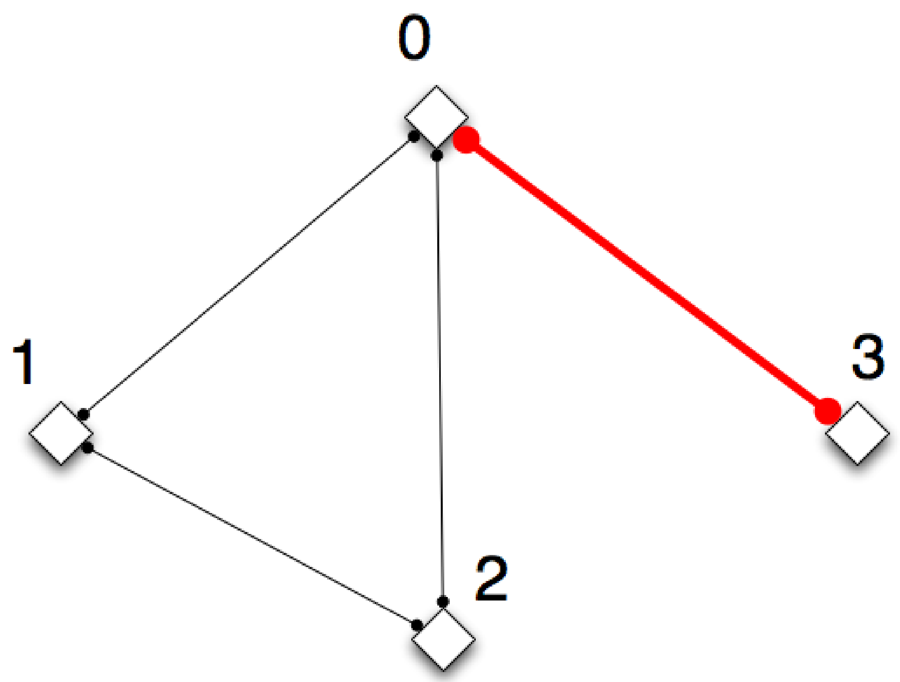
Edge Polytopes



Incidence Matrix of G

1	1	1	0
1	0	0	1
0	1	0	1
0	0	1	0

Edge Polytopes



Incidence Matrix of G

1	1	1	0
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0	1	0	1
0	0	1	0

Edge Polytopes (2)

Incidence Matrix of G

1	1	1	0
1	0	0	1
0	1	0	1
0	0	1	0

$\mathscr{P}$ = convex hull of

{(1, 1, 0, 0),

(1, 0, 1, 0),

(1, 0, 0, 1),

(0, 1, 1, 0)}

Edge Polytopes (2)

Incidence Matrix of G

1	1	1	0
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$\mathscr{P}$ = convex hull of

{(1, 1, 0, 0),

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We compute the roots of Ehrhart polynomial of edge polytopes for all connected simple graphs of order up to 9.

Computing Steps

1. Generate graphs
2. Obtain the Ehrhart polynomial for each edge polytope for a graph
3. Find the roots of the polynomial

Generating all connected simple graphs

The outline is:

- Generating connected simple graphs of order d from such graphs of order d-1, adjoining the least degree vertex.
- Connecting non-connected subgraphs

  (for $d \geqq 9$)

Generating all connected simple graphs (2)

Finally,

- Pick up one representative for each class of isomorphic graphs

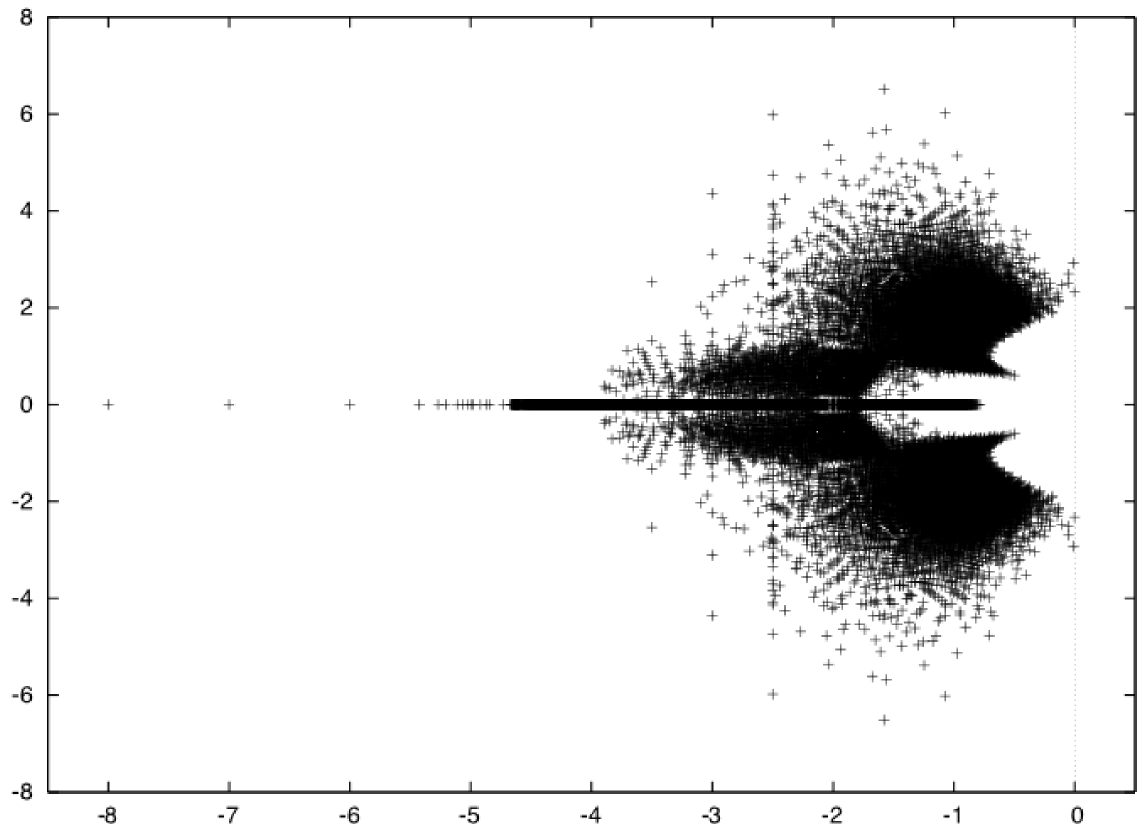
The number of connected simple graphs

d	4	5	6	7	8	9	10
$\|\{G\}\|$	6	21	112	853	11117	261080	11716571

Computing Steps (again)

1. Generate graphs
2. Obtain the Ehrhart polynomial for each edge polytope for a graph
3. Find the roots of the polynomial

The roots of Ehrhart polynomials of edge polytopes of graphs of order 9



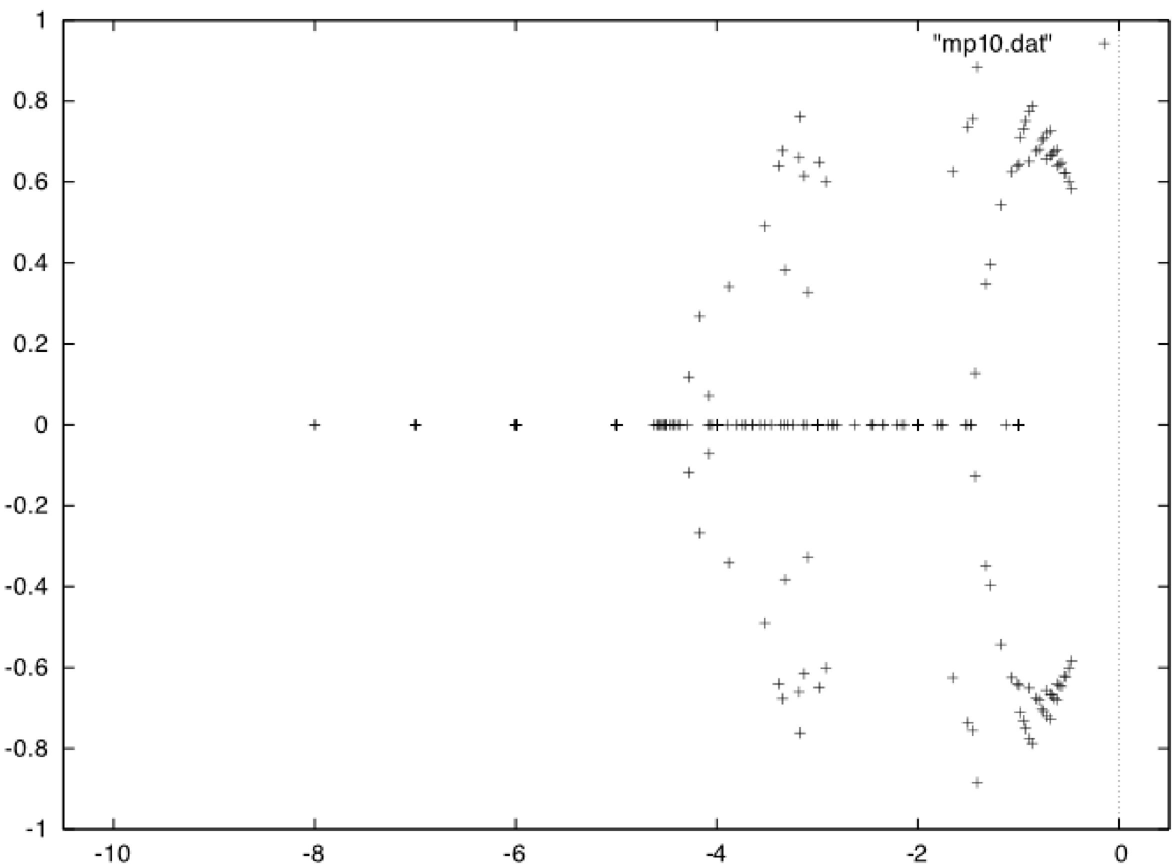
Complete multi-partite graphs

Complete multi-partite graphs is a special subclass of connected simple graphs.
Their Ehrhart polynomials are given explicitly:

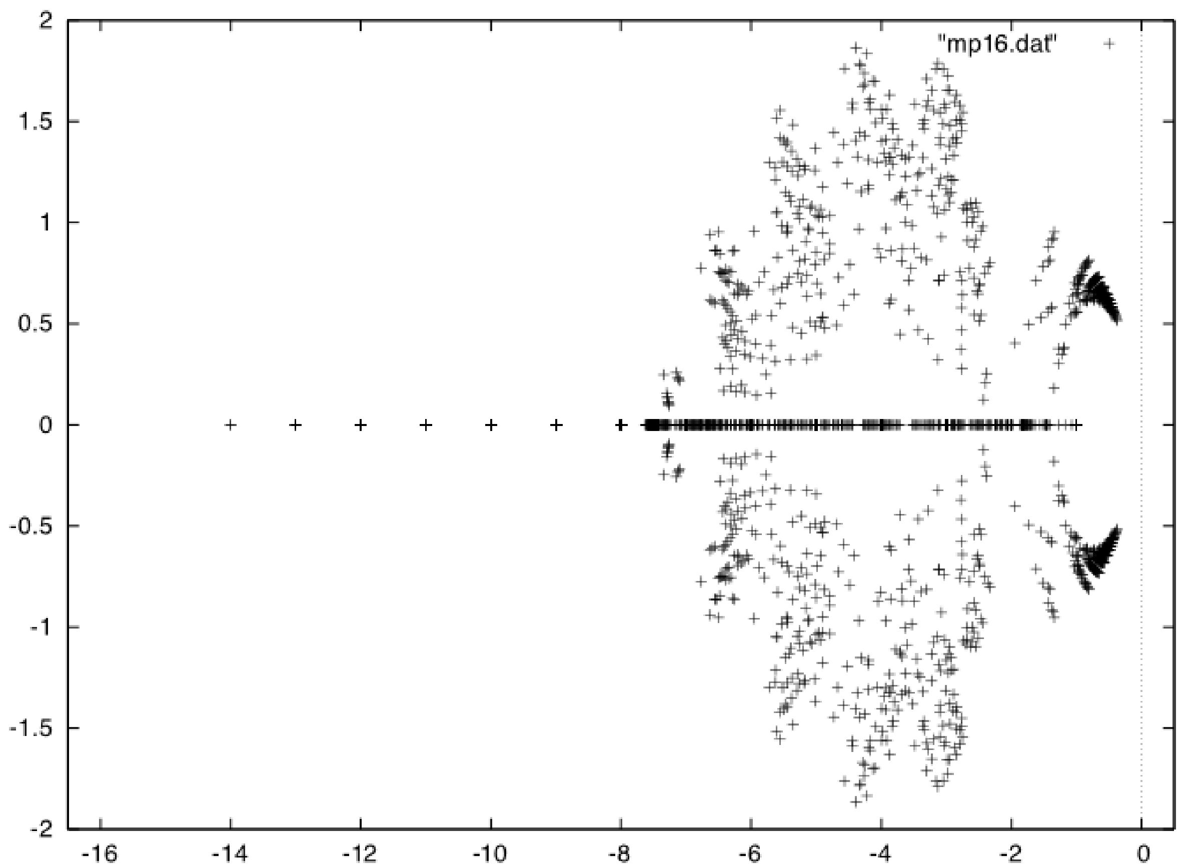
$$\binom{d+2\mathrm{m}-1}{d-1} - \sum_{k=1}^{t} \sum_{1 \le i \le j \le q_k} \binom{j-i+m-1}{j-i} \binom{d-j+m-1}{d-j}$$

We obtained the root of Ehrhart polynomial of the complete multi-partite graphs of order up to 22, by the formula above.

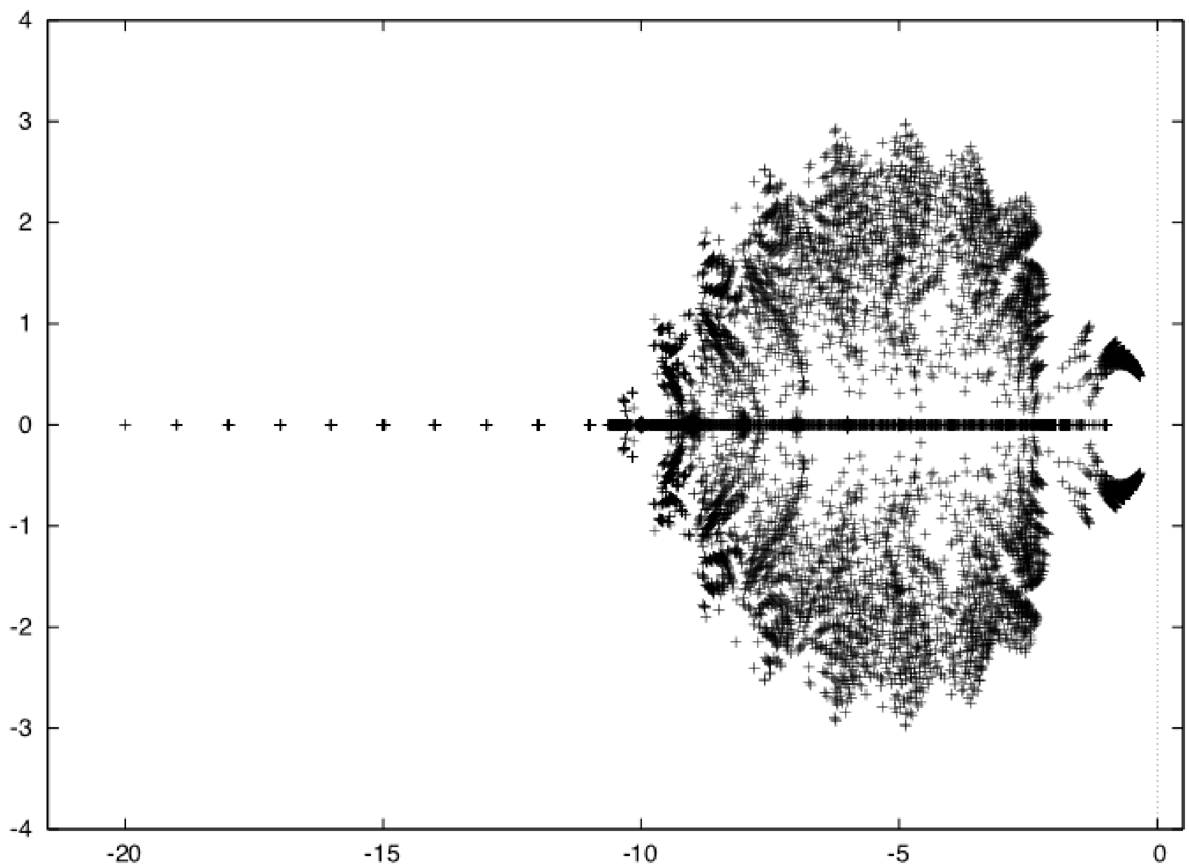
The root distribution for complete multi-partite graphs of order 10



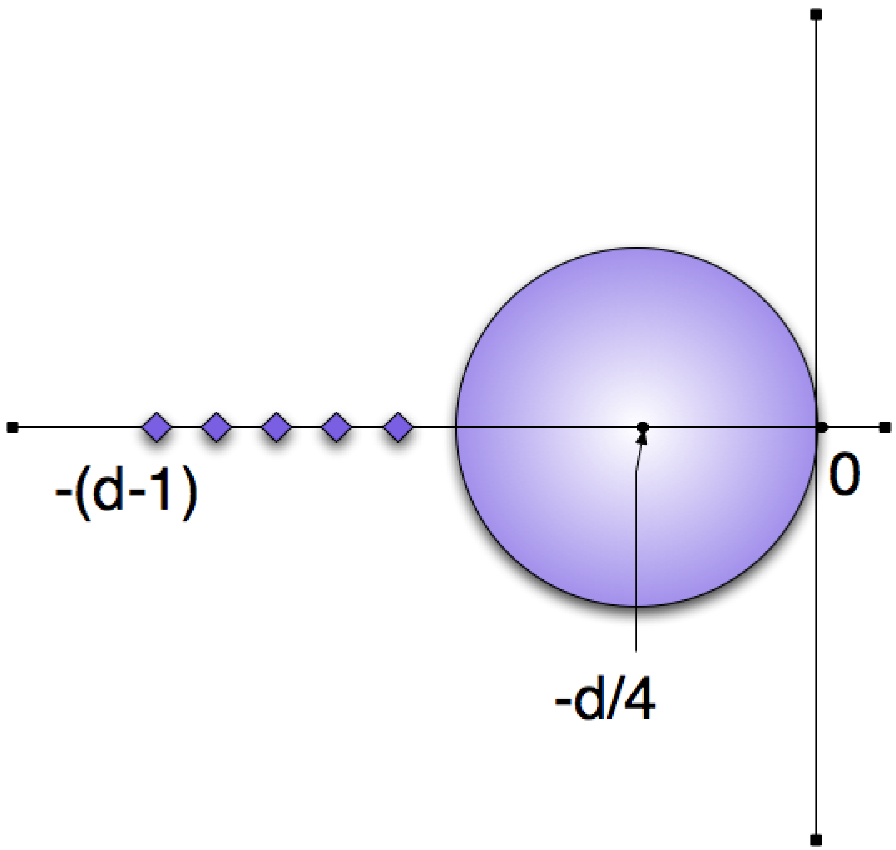
The root distribution for complete multi-partite graphs of order 16



The root distribution for complete multi-partite graphs of order 22



Conjecture 1



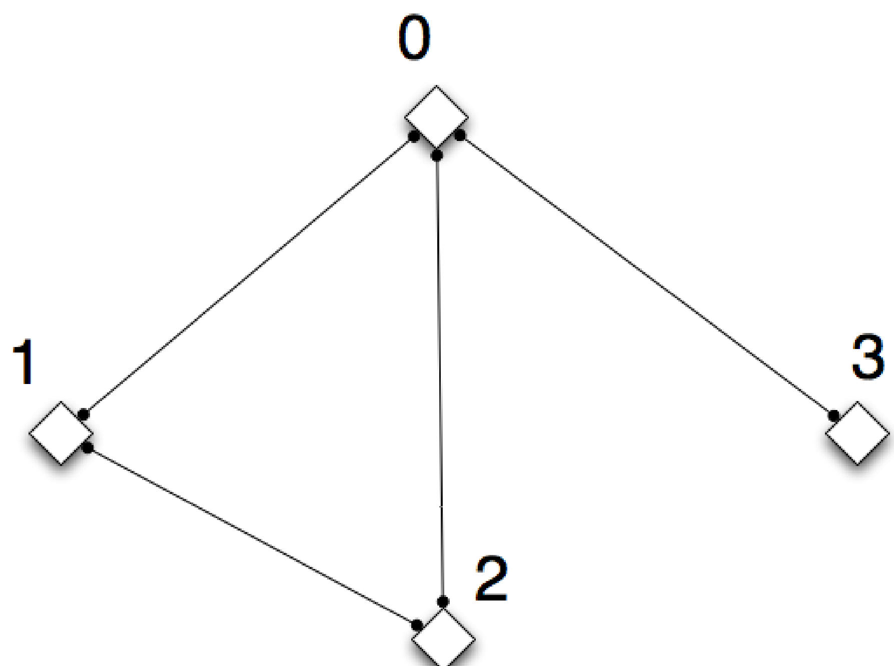
For any d greater than or equal to 3, all the roots of Ehrhart polynomials for complete multi-partite graphs are in or on a circle centered at $-d/4$ with diameter $d/4$, or in negative integers $-1, -2, \ldots -(d-1)$.

Examples

There are a few cases shown to satisfy the conjecture 1.

- Complete bipartite graphs
- $K_{n,1,1}$

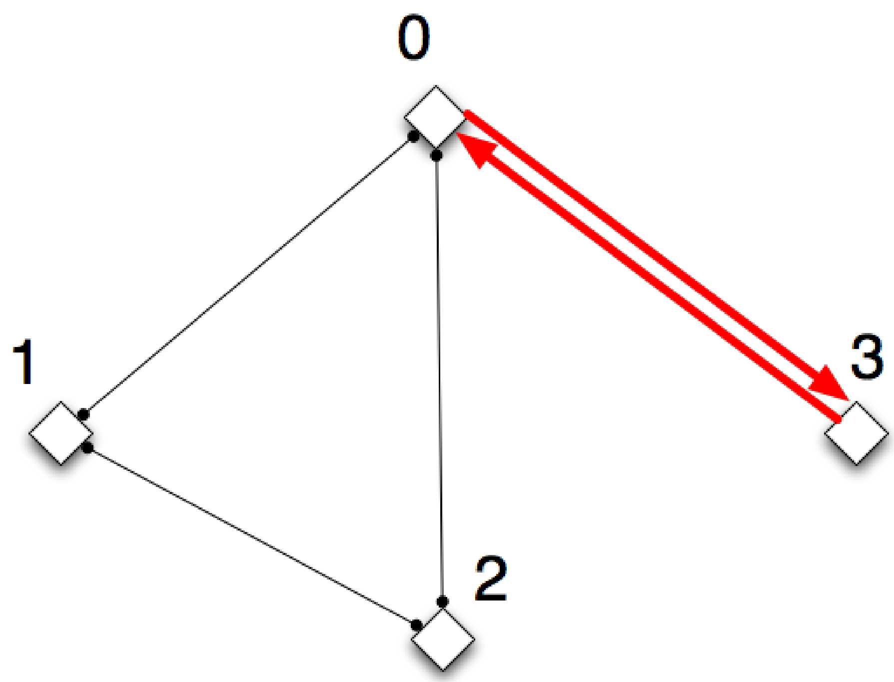
Symmetric Edge Polytopes



Incidence Matrix of G as symmetric directed graph

1	-1	...	1	-1	...
-1	1	...	0	0	...
0	0	...	0	0	...
0	0	...	-1	1	

Symmetric Edge Polytopes



Incidence Matrix of G as symmetric directed graph

1	-1	...	1	-1	...
-1	1	...	0	0	...
0	0	...	0	0	...
0	0	...	-1	1	

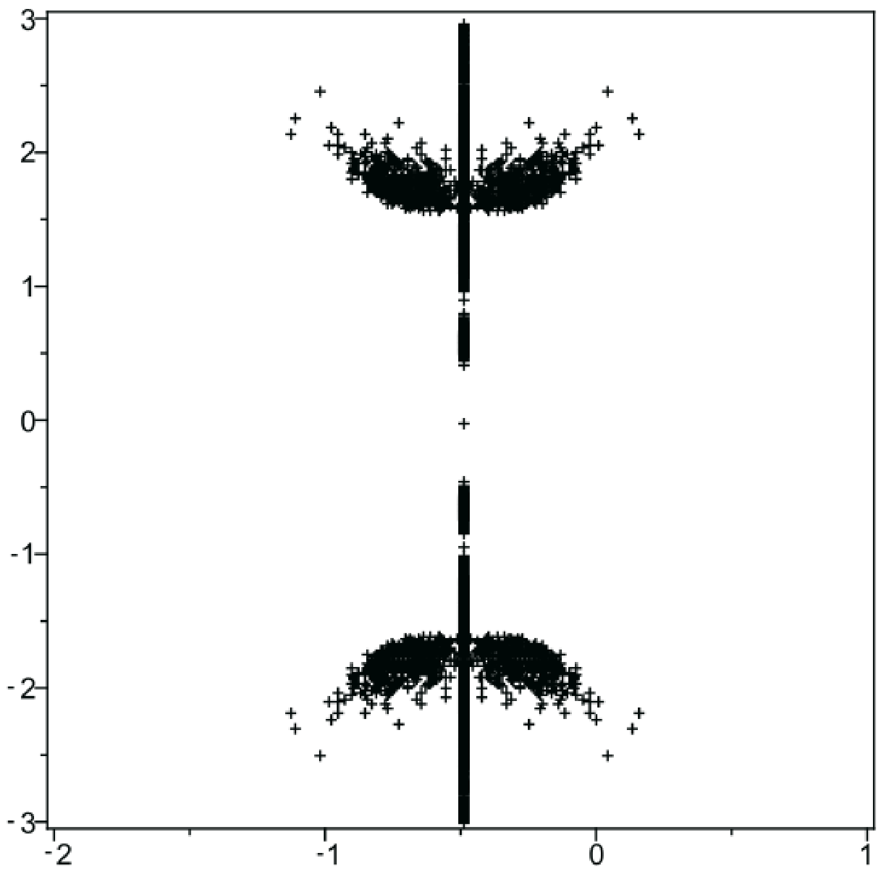
Proposition Let P be a symmetric edge polytope of a graph G. Then P is a terminal Gorenstein Fano polytope with dimension (d - 1).

Theorem Any centrally symmetric smooth Fano polytope is unimodular equivalent with the symmetric edge polytope of a graph with no even cycle.

The number of non equivalent graphs

d	3	4	5	6	7	8
c.s.g.	2	6	21	112	853	11117
n.e.	2	5	16	75	560	7772

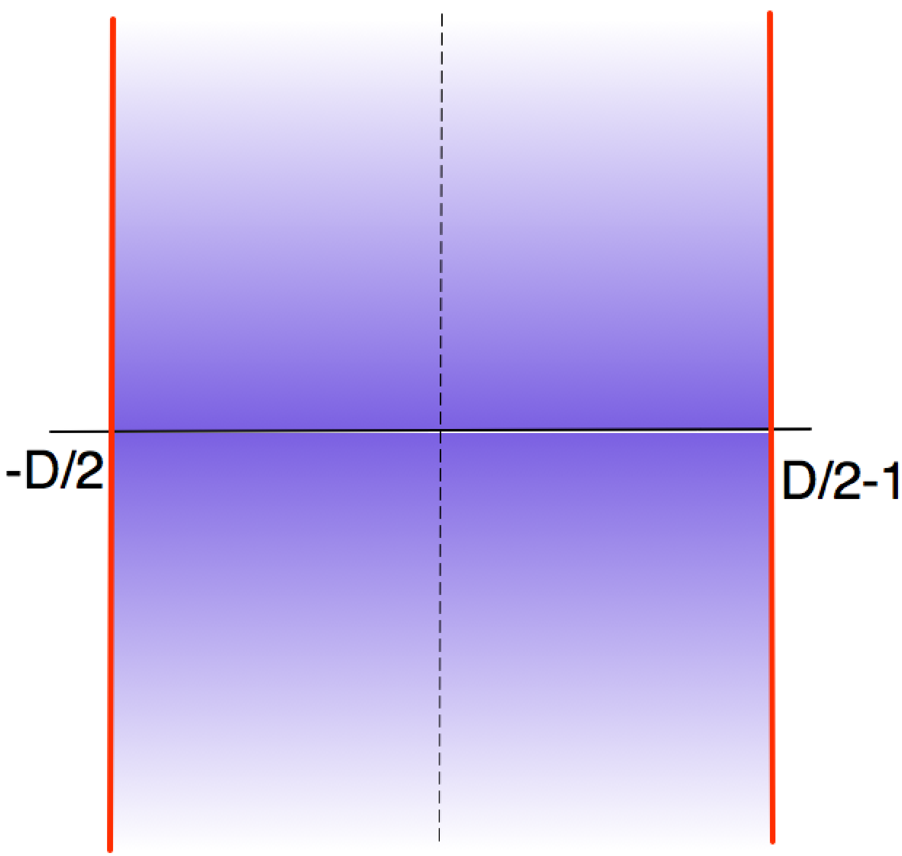
Roots distribution for symmetric edge polytopes of graphs of order 8



Roots distribute symmetric with respect to the line:

$$Re(z) = -1/2$$

Conjecture 2



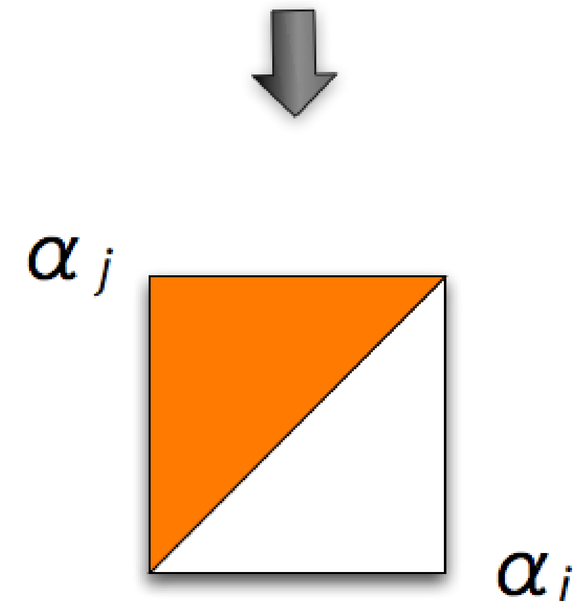
All roots α of Ehrhart polynomials of symmetric edge polytopes with dimension D have their real part in the range:

$$-D/2 \leqq \mathrm{Re}(\alpha) \leqq D/2-1$$

Order Polytopes

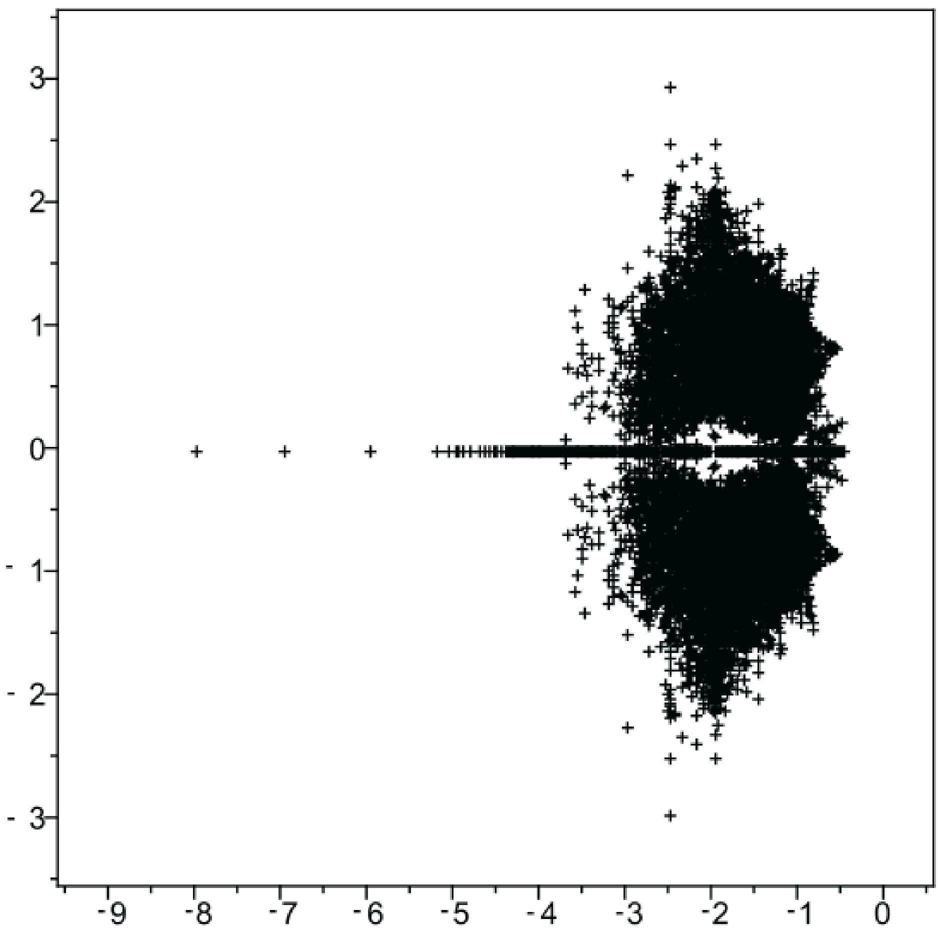
For a finite poset P of d elements, order polytope is defined in a unit cube in d-dimensional real space as ...

$y_i \leqq y_j$



$\alpha_i \leqq \alpha_j$

Roots distribution for order polytopes with order up to 8



Summary

- We computed roots of Ehrhart polynomials of three kind of polytopes arising from graphs and posets.
- We make two conjectures about distribution of the roots.

For the details, please take a look at arXiv:1003.5444