On the relation of depth modulo a graded ideal and its initial ideal

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K: an infinite field
$S = K[X_1, \ldots, X_r]$: a polynomial ring
We assume that S is graded by a weight vector $w = (w_1, \ldots, w_r) \in (\boldsymbol{N} \setminus \{0\})^r$, that is $\deg X_i = w_i$ for $i = 1, \ldots, r$.
I: a graded ideal of S

Definition 1 The Krull dimension $\mathrm{Krulldim} S/I$ of S/I is $\max\{d \mid \exists P_0, P_1, \ldots, P_d$ such that $I \subset P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$ and P_i is a prime ideal for any $i\}$.

Fact 2 $\mathrm{Krulldim} S/I = \max\{t \mid \exists i_1, \ldots, i_t$ *such that the image of* $X_{i_1}, \ldots, X_{i_t}$ *in* S/I *are algebraically independent over* $K\}$.

Definition 3 $\operatorname{depth} S/I := \min\{i \mid \operatorname{Ext}^i_S(K, S/I) \neq 0\}$

Fact 4 $\operatorname{depth} S/I = \min\{i \mid H^i_{\mathfrak{m}}(S/I) \neq 0\}$, *where* $\mathfrak{m} = (X_1, X_2, \ldots, X_r)$*.*

Fact 5 $\operatorname{depth} S/I \leq \operatorname{Krulldim} S/I$*.*

Definition 6 If $\operatorname{depth} S/I = \operatorname{Krulldim} S/I$, we say that S/I is Cohen-Macaulay.

Theorem 7 (Auslander-Buchsbaum) $\operatorname{depth} S/I = r - \operatorname{projdim} S/I$*.*

In our situation,

Fact 8 $\operatorname{projdim} S/I = \max\{i \mid \operatorname{Tor}^S_i(K, S/I) \neq 0\}$*.*

Definition 9 $\beta_{ij} := \dim_K \operatorname{Tor}^S_i(K, S/I)_j$. β_{ij} are called Betti numbers.

Now assume that a monomial order $<$ on S is defined.
J: a graded ideal of S.

Fact 10 $\mathrm{Krulldim} S/\operatorname{in}(J) = \mathrm{Krulldim} S/J$.

Fact 11 *Let T be a new variable. There is an ideal $\tilde{J}$ in $S[T] = K[T][X_1, \ldots, X_r]$ such that $S[T]/\tilde{J}$ is flat over $K[T]$, $S[T]/((T) + \tilde{J}) \simeq S/\operatorname{in}(J)$ and $S[T]/((T - u) + \tilde{J}) \simeq S/J$ for any $u \in K \setminus \{0\}$.*

I.e., if we substitue T by u in $S[T]$, then $S[T]/\tilde{J}$ is isomorphic to S/J if $u \neq 0$ and is isomorphic to $S/\operatorname{in}(J)$ if $u = 0$.

Corollary 12 *Betti numbers are upper semi-continuous, i.e., for any i, j and for any $a \in \boldsymbol{R}$, $\{u \in K \mid \beta_{ij}(u) \in [a, \infty)\}$ is a closed subset of K (in the Zariski topology).*

Corollary 13 $\mathrm{depth} S/\operatorname{in}(J) \leq \mathrm{depth} S/J$.

Example 14 Let n be an integer with $n > 2$, $X = (X_{ij})$ an $n \times n$ symmetric matrix of indeterminates, i.e., $\{X_{ij}\}_{1 \leq i \leq j \leq n}$ is a family of independent indeterminates and $X_{ji} = X_{ij}$ for $i < j$. Set $S = K[X_{ij} \mid 1 \leq i \leq j \leq n]$ with $\deg X_{ij} = 1$ for any i and j and consider the degree reverse lexicographic order given by $X_{11} > X_{12} > \cdots > X_{1n} > X_{22} > X_{23} > \cdots > X_{nn}$. Let $J = I_2(X)$ be the ideal generated by 2-minors of X. Then $\operatorname{depth} S/J = n$ whereas $\operatorname{depth} S/\operatorname{in}(J) = 2$.

Theorem 15 *Assume that* $S/\operatorname{in}(J)$ *is reduced and has finite local cohomologies, i.e.* $\dim_K H^i_{\mathfrak{m}}(S/\operatorname{in}(J))$ *is finite for* $i < \mathrm{Krulldim} S/\operatorname{in}(J)$*. Then* $\mathrm{depth} S/\operatorname{in}(J) = \mathrm{depth} S/J$*. In particular, if* S/J *is Cohen-Macaulay, then so is* $S/\operatorname{in}(J)$*.*

Remark 16 In the situation of Example 14, X_{ij}^2 is a member of the minimal generationg system of $\operatorname{in}(J)$ for any i, j with $i < j$. In particular $S/\operatorname{in}(J)$ is not reduced. On the other hand, if we consider the lexicographic order or the degree lexicographic order on $K[X_{ij} \mid 1 \leq i \leq j \leq n]$, then $S/\operatorname{in}(J)$ is Cohen-Macaulay of Krull dimension n.