CMC-Labの使い方

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(1) ◦ Basic examples

(2) ◦ Dorfmeister-Pedit-Wu method

(3) ◦ Further examples and explanations

(4) ◦ Algorithms for CMC-Lab

1 Installation of CMC-Lab

Fact : CMC-Lab was constructed by Nicolas Schmitt for research of constant mean curvature surfaces.

(1) Linux version
$\Rightarrow$ KNOPPIX/Math 2006

(2) Java version
$\Rightarrow$ http://tmugs.math.metro-u.ac.jp/javacmclab030926.zip

2 Dorfmeister-Pedit-Wu method

Isomorphism of vector spaces:

$$\mathbb{R}^3 \Longleftrightarrow su(2) = \left\{ A \in \mathsf{Mat}(2, \mathbb{C}) \ ; \ \ \bar{A}^t = -A \right\} \quad ,$$

Adjoint group actions on $su(2)$ by $SU(2) \overset{2:1}{\Longleftrightarrow}$ Rotations of $\mathbb{R}^3$ by $SO(3)$

Generalized Weierstrass Representation

(Dorfmeister-Pedit-Wu, 1998)

Step1 :

- $\eta(z,\lambda) = \sum_{n=-1}^{\infty} A_n \lambda^n dz.$
- 2×2 matrix differential form.
- diagonal even in λ, off-diagonal odd in λ.
- A_j are holomorphic with respect to z.
- $\det A_{-1} \neq 0.$

Step2 : Solve the ODE $dC = C\eta$.

Step3 : Iwasawa decomposition: $C = FW_+$

- $F = F(z, \bar{z}, \lambda)$ is unitary for all z $\in \mathfrak{D}, \lambda \in \mathbb{S}^1$
- $W_+ = \sum_{n=0}^{\infty} W_{n,+}\lambda^n$.

Step4 : (Sym-Bobenko-Formula)

$$\Psi_\lambda(z) = -\frac{1}{2H}\left\{\left(i\lambda\frac{d}{d\lambda}F\right)F^{-1} + F\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}F^{-1}\right\}$$

$\Rightarrow \begin{cases} \Psi_\lambda \text{ is a CMC-immersion from } \mathfrak{D} \text{ to } \mathbb{R}^3. \\ \text{Every CMC-immersion can be obtained this way.} \end{cases}$

3 Algorithm for CMC-Lab

Algorithm for Step3

Lemma 1 Let $C \in \Lambda SL_2(\mathbb{C})$ and C^1, C^2 be the columns of C. If $x, y \in W \cap (\lambda W)^\perp$, then

$$\langle x, y \rangle_{\mathbb{C}^2} = \langle x, y \rangle_H \text{ and } \dim(W \cap (\lambda W)^\perp) = 2 \ ,$$

where

$$\langle x, y \rangle_H = \frac{1}{2\pi i} \int_{C_r} \langle x, y \rangle_{\mathbb{C}^2} \frac{d\lambda}{\lambda} \ .$$

Theorem 2

$$W = \mathrm{span}\{C^1, \lambda C^1, \cdots, C^2, \lambda C^2, \cdots\} \ .$$

Set

$$P^j : C^j \to \lambda W \ \ \text{(projection to } \lambda W\text{)} \ .$$

and

$$P = (P^1, P^2) \ .$$

Then $P = CB_+$ for some loop B_+ with positive Fourier terms. Set

$$G = (G^1, G^2) = C - P \ \ .$$

Take unitary part of G via Hilbert norm, that is, $G = FB_0$

$$B_0 = \begin{pmatrix} |G^1| & \langle G^2, G^1/|G^1|\rangle \\ 0 & |G^2 - G^1/|G^1|\langle G^2, G^1/|G^1|\rangle| \end{pmatrix} \ \ .$$

Then $C = F \cdot B_0(I - B_+)^{-1}$ is the Iwasawa decomposition of C.

Problem : Find the projection P!

Proposition 3

$$\mathcal{A} = \{a_1, \cdots, a_n\} \text{ : a basis for } \mathbb{C}^n.$$

Take $0 \leq r \leq n$,

$$p : \mathbb{C}^n \to \mathbb{C}^n \text{ : projection to the subspace } \{a_1, \cdots, a_r\},$$

$$A = (a_1, \cdots, a_n) \in \mathrm{GL}_n(\mathbb{C}) \quad ,$$

and

$$\tilde{P} = \begin{pmatrix} I_r & 0 \\ 0 & O_{n-r} \end{pmatrix} \in M_{n \times n}(\mathbb{C}) \ .$$

Then p can be written as follows:

1 $A\tilde{P}A^{-1}$,

2 $U\tilde{P}\bar{U}^t$,
where $A = UT$ is the QR-decomposition of A.

Algorithm : Take a loop

$$A = \begin{pmatrix} \sum_{k=-n}^{n} a_k^{11}\lambda^k & \sum_{k=-n}^{n} a_k^{12}\lambda^k \\ \sum_{k=-n}^{n} a_k^{21}\lambda^k & \sum_{k=-n}^{n} a_k^{22}\lambda^k \end{pmatrix} \in \Lambda SL_2(\mathbb{C}) \quad .$$

Set r is even, $r/2 \leq n$,

$$a_1 = \begin{pmatrix} \sum_{k=-n}^{n} a_k^{11}\lambda^k \\ \sum_{k=-n}^{n} a_k^{21}\lambda^k \end{pmatrix} \quad , \quad a_2 = \begin{pmatrix} \sum_{k=-n}^{n} a_k^{12}\lambda^k \\ \sum_{k=-n}^{n} a_k^{22}\lambda^k \end{pmatrix} \quad .$$

and

$$\lambda W = \mathrm{span}\left\{\lambda a_1, \lambda^2 a_1, \cdots, \lambda^{r/2} a_1, \lambda a_2, \cdots, \lambda^{r/2} a_2\right\} \quad .$$

Then the projection p can be computed by Proposition 3 as follows:

$$(U_0, 0)\tilde{P}\overline{(U_0, 0)}^t,$$

where $(A_0, 0) = (U_0, 0)\begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix}$ is QR-decomposition of A_0.

$$
A_0 = \left(\begin{array}{cccc|cccc}
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
a^{11}_{-n} & 0 & \cdots & \vdots & a^{12}_{-n} & 0 & \cdots & \vdots \\
\vdots & a^{11}_{-n} & \ddots & \vdots & \vdots & a^{12}_{-n} & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & 0 \\
\vdots & \vdots & & a^{11}_{-n} & \vdots & \vdots & & a^{12}_{-n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a^{11}_{n-1} & a^{11}_{n-2} & \cdots & a^{11}_{n-r/2} & a^{12}_{n-1} & a^{12}_{n-2} & \cdots & a^{12}_{n-r/2} \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
a^{21}_{-n} & 0 & \cdots & \vdots & a^{22}_{-n} & 0 & \cdots & \vdots \\
\vdots & a^{21}_{-n} & \ddots & \vdots & \vdots & a^{22}_{-n} & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & 0 \\
\vdots & \vdots & & a^{21}_{-n} & \vdots & \vdots & & a^{22}_{-n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a^{21} & a^{21} & \cdots & a^{21} & a^{22} & a^{22} & \cdots & a^{22}
\end{array}\right)
$$

1 N. Schmitt, CMC surfaces and related videos,
http://www.mathematik.uni-tuebingen.de/ab/Differentialgeometrie/gallery/

2 J. Dorfmeister, F. Pedit, H. Wu, Weierstrass type representation of harmonic maps into symmetric spaces, Comm. Anal. Geom. **(6)4**, (1998), 633–668.

3 S. Fujimori, S-P. Kobayashi, W. Rossman, A basic introduction to loop group methods for constant mean curvature surfaces (http://www.math.sci.kobe-u.ac.jp/HOME/wayne/wayne-e.html).