Triangular spiral tilings and origami

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Spiral phyllotaxis

Phyllotaxis is the arrangement of leaves and other organs of plants.

Spiral phyllotaxis: "Sunflower" and "Pine cone" and so on.

The golden section:

$$\tau = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cdots}}.$$

The fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \cdots .$$

The lucas sequence:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \cdots .$$

Figure: **Capitulum of sunflower**

counterclockwise: 89 spirals
clockwise: 55 spirals

Mathematical studies of phyllotaxis

Mathematical studies of phyllotaxis were started by people such as **Bravais brothers** in the first half of 19th century.

■ **Comprehensive text of phyllotaxis**:

1994: Roger.V.Jean, Phyllotaxis, A Systemic Study in Plant Morphogenesis, Cambrige University Press.

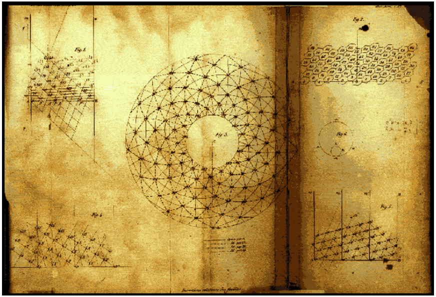

Figure: **Bravais's sketch ?** (http://www.math.smith.edu/phyllo: PHYLLOTAXIS)

■ **Experiment**:

1996: S.Douady and Y.Couder, Phyllotaxis as a Dynamical Self Organizing Process, J. Theor. Biol, 178, 255-274, 275-294, 295-312.

■ **Bifurcation structure of a dynamical system**:

2002: P.Atela, C.Gole and S.Hotton, A dynamical system for plant pattern formation: A rigorous analysis, J.Nonlinear Sci. Vol. 12, pp. 641-676.

■ **Mathematical models**:

2006: R.S.Smith at.el, A plausible model of phyllotaxis, Proc. Nat. Acad. Sci. 103 (5) 1301-1306.

2012: Y.Tanaka and M.Mimura, Reaction-diffusion model for inflorescence of sunflower, The 3rd Taiwan-Japan joint workshop for young scholars in applied mathematics, National Taiwan University.

Mathematical studies of spiral tilings

2008: A.Hizume, Fibonacci Tornado, in: Proceedings of the 11th Bridges Conference, 485-486.

2009: A.Hizume and Y.Yamagishi, Real Tornado, in: Proceedings of the 12th Bridges Conference, 239-242.

2012: T.Sushida, A.Hizume and Y.Yamagishi, Triangular spiral tilings, J.Phys.A: Math.Theor. 45, 23, 235203.

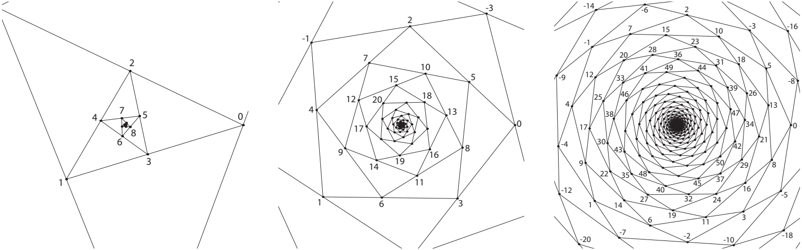

Figure: **Triangular spiral tilings with phyllotactic patterns**

Quadrilateral spiral multiple tilings

Let $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{D}\backslash\mathbb{R}$. Let $m, n > 0$ be relatively prime integers.

We denote a point with the complex coordinate ζ^j, $j \in \mathbb{Z}$ by A_j.

We consider a spiral lattice $\mathcal{S} = \{\mathrm{A}_j\}_{j\in\mathbb{Z}}$.

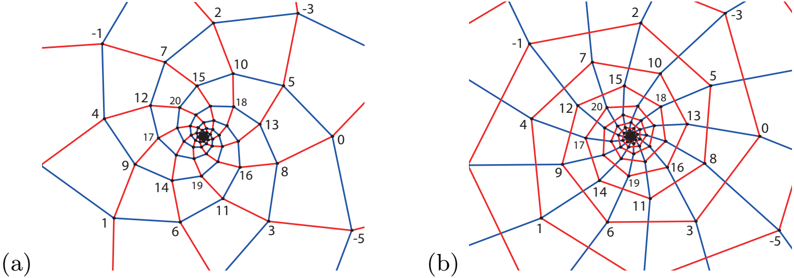

Figure: **Quadrilateral spiral multiple tilings:**
Each j denotes a position of a point A_j, $j \in \mathbb{Z}$.
$r = 0.94$, $\theta = 2\pi\tau$, $\tau = \frac{1+\sqrt{5}}{2}$, (a) $(m, n) = (5, 8)$, (b) $(m, n) = (13, 3)$

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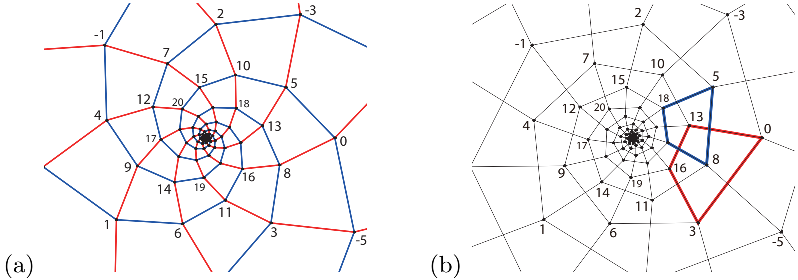

Figure: **Quadrilateral spiral multiple tilings:**
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Definition 1

A tiling of a two dimensional manifold X is a family $\mathcal{T} = \{T_j\}_j$ of topological disks $T_j \subset X$ which satisfies the following conditions:

$$X = \bigcup_j T_j, \quad \mathrm{int}(T_j) \cap \mathrm{int}(T_k) = \emptyset \quad (j \neq k).$$

Each T_j is called a tile.

Let $v \neq 0$ be an integer. Let $C_v := \mathbb{C}/2\pi v\sqrt{-1}\mathbb{Z}$ be a cylinder.

We denote $\mathbb{C}$ which punctures at the origin O by $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

By the exponential map $\exp : C_v \to \mathbb{C}^*$, $w + 2\pi v\sqrt{-1}\mathbb{Z} \to z = e^w$,

C_v is a covering space of $\mathbb{C}^*$ of a degree $|v|$.

Let $\mathcal{T}' = \left\{T'_j\right\}_j$ be a tiling of C_v. Then $\exp\left(\mathcal{T}'\right) = \left\{\exp\left(T'_j\right)\right\}_{T'_j \in \mathcal{T}'}$ is a multiple tiling of $\mathbb{C}^*$ of multiplicity $|v|$.

Definition 2

Let V' be an additive subgroup of C_v.
We say that $\mathcal{T}'$ admits a transitive action by V' if

(i) $\forall T' \in \mathcal{T}'$, $\forall \eta \in V'$, $T' + \eta \in \mathcal{T}'$ **and**

(ii) $\forall T'_1, T'_2 \in \mathcal{T}'$, $\exists \eta \in V'$ **such that** $T'_2 = T'_1 + \eta$.

If $\mathcal{T}'$ admits a transitive action by an additive subgroup $\xi\mathbb{Z}$ which is generated by a single element $\xi \in C_v$,

then $\mathcal{T} = \exp(\mathcal{T}')$ is called a spiral multiple tiling of multiplicity $|v|$.

Let $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{D}\backslash\mathbb{R}$. Let $m, n > 0$ be relatively prime integers.

We denote the principal argument of $z \in \mathbb{C}^*$ by $-\pi < \arg(z) \leq \pi$.

We suppose that $T_0 := \square \mathrm{A}_0\mathrm{A}_m\mathrm{A}_{m+n}\mathrm{A}_n$ is a quadrilateral in $\mathbb{C}^*$ in this order of vertices.

Let

$$\xi_m = m\log(r) + \sqrt{-1}\left(m\theta - 2\pi[[\frac{m\theta}{2\pi}]]\right) \in \log(\zeta^m),$$

$$\xi_n = n\log(r) + \sqrt{-1}\left(n\theta - 2\pi[[\frac{n\theta}{2\pi}]]\right) \in \log(\zeta^n),$$

where $[[x]] \in \mathbb{Z}$ is an integer which is the nearest to $x \in \mathbb{R}$ such that $-\frac{1}{2} < \langle x \rangle := x - [[x]] \leq \frac{1}{2}$.

We obtain a tiling $\mathcal{T}' := \{T_0' + k_1\xi_m + k_2\xi_n\}_{k_1,k_2\in\mathbb{Z}}$ of C_v.

where T_0' is a component of $\log(T_0)$ which has 0, ξ_m, $\xi_m + \xi_n$ and ξ_n on its boundary.

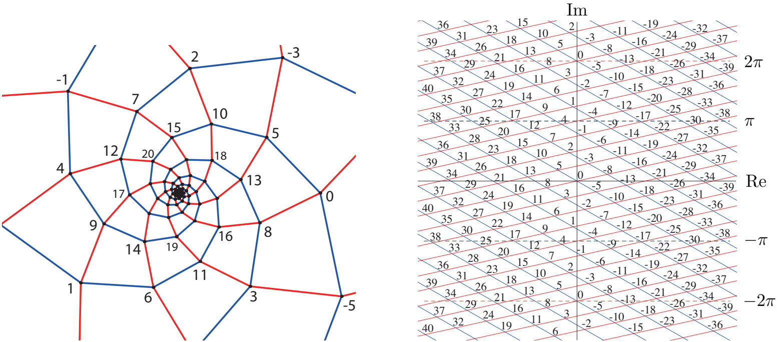
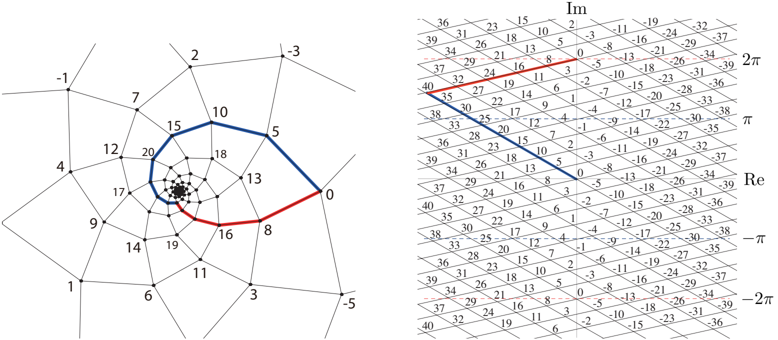


Figure: **A quadrilateral spiral tiling and a tiling of C_v, $v = 1$: $r = 0.94$, $\theta = 2\pi\tau$, $\tau = \frac{1+\sqrt{5}}{2}$, $(m, n) = (5, 8)$**

In the right figure, each $j \in \mathbb{Z}$ denotes a position of

$$\xi_j = j \log(r) + \sqrt{-1}(j(\theta + 2\pi\ell) + 2\pi k v), \ k \in \mathbb{Z}, \ \ell \in \mathbb{Z}$$



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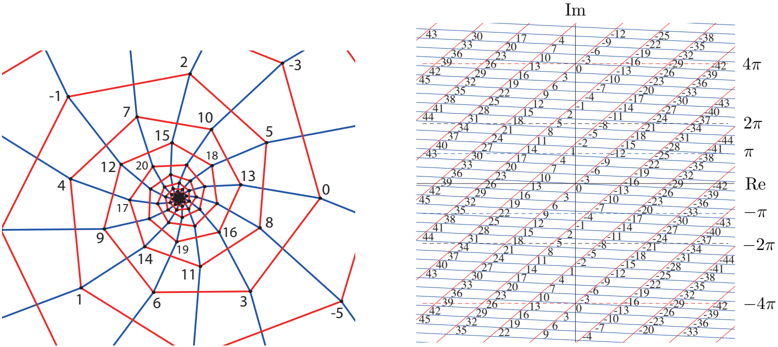
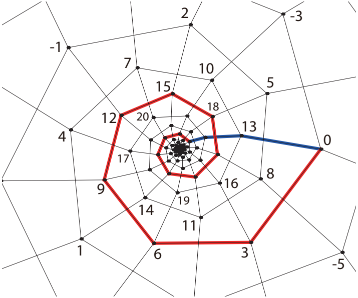


Figure: **A quadrilateral spiral multiple tiling and a tiling of C_v, $v = 2$: $r = 0.94$, $\theta = 2\pi\tau$, $\tau = \frac{1+\sqrt{5}}{2}$, $(m, n) = (13, 3)$**

In the right figure, each $j \in \mathbb{Z}$ denotes a position of

$$\xi_j = j \log(r) + \sqrt{-1}(j(\theta + 2\pi\ell) + 2\pi k v),\ \ k \in \mathbb{Z},\ \ \ell = 2\ell' + 1,\ \ \ell' \in \mathbb{Z}$$



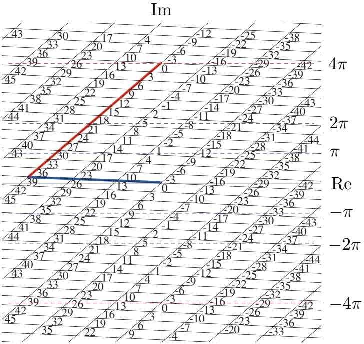


Figure: **A quadrilateral spiral multiple tiling and a tiling of C_v, $v = 2$: $r = 0.94$, $\theta = 2\pi\tau$, $\tau = \frac{1+\sqrt{5}}{2}$, $(m, n) = (13, 3)$**

In the right figure, each $j \in \mathbb{Z}$ denotes a position of

$$\xi_j = j \log(r) + \sqrt{-1}(j(\theta + 2\pi\ell) + 2\pi kv), \ \ k \in \mathbb{Z}, \ \ \ell = 2\ell' + 1, \ \ \ell' \in \mathbb{Z}$$

Let $a, b \in \mathbb{Z}$ be integers such that $bm - an = 1$.

Then we obtain the followings:

$$\xi := b\xi_m - a\xi_n = \log(r) + \sqrt{-1}(\theta + 2\pi\ell) \in \log(\zeta), \quad \ell = a[[\frac{n\theta}{2\pi}]] - b[[\frac{m\theta}{2\pi}]].$$

$$v := m[[\frac{n\theta}{2\pi}]] - n[[\frac{m\theta}{2\pi}]] = \frac{1}{2\pi}(n\arg(\zeta^m) - m\arg(\zeta^n)).$$

In C_v, we have $[[\frac{n\theta}{2\pi}]]\xi_m - [[\frac{m\theta}{2\pi}]]\xi_n \equiv 0$,

$$m\xi \equiv \xi_m, \ n\xi \equiv \xi_n \text{ and } \xi_m\mathbb{Z} + \xi_n\mathbb{Z} \equiv \xi\mathbb{Z} \bmod 2\pi v\sqrt{-1}\mathbb{Z}.$$

Let $\mathcal{T}' = \{T_0' + k_1\xi_m + k_2\xi_n\}_{k_1,k_2\in\mathbb{Z}} = \{T_0' + k\xi\}_{k\in\mathbb{Z}}$.

Then $\mathcal{T}'$ is a tiling of C_v that admits a transitive action by $\xi\mathbb{Z}$ and we have $\mathcal{T} = \exp(\mathcal{T}')$.

Let $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{D}\backslash\mathbb{R}$. Let $m, n > 0$ be relatively prime integers.

We denote $\mathbb{C}$ which punctures at the origin O by $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$.

We denote the principal argument of $z \in \mathbb{C}^*$ by $-\pi < \arg(z) \leq \pi$.

Proposition 3

Suppose that $\zeta^m, \zeta^n \notin \mathbb{R}_-$.
If $T_0 := \square \mathrm{A}_0\mathrm{A}_m\mathrm{A}_{m+n}\mathrm{A}_n$ is a quadrilateral in $\mathbb{C}^*$ in this order of vertices, then a family of quadrilaterals

$$\mathcal{T} = \{T_j := \square \mathrm{A}_j\mathrm{A}_{j+m}\mathrm{A}_{j+m+n}\mathrm{A}_{j+n}\}_{j\in\mathbb{Z}} \tag{1}$$

is a quadrilateral spiral multiple tiling of multiplicity $|v|$, where v is given by

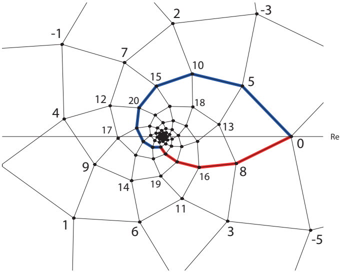
$$v = \frac{1}{2\pi}(n\arg(\zeta^m) - m\arg(\zeta^n)). \tag{2}$$

In (1), a combinational index (m, n) is called a parastichy pair.

Definition 4

(i) **A parastichy pair** (m, n) **is an opposed parastichy pair if** $(\arg(\zeta^m))(\arg(\zeta^n)) < 0$**.**

(ii) **A parastichy pair** (m, n) **is a non-opposed parastichy pair if** $(\arg(\zeta^m))(\arg(\zeta^n)) > 0$**.**

(a) 

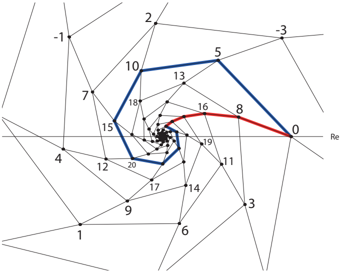
(b) 

Figure: $r = 0.94$, $(m, n) = (5, 8)$, (a) $\theta = 2\pi\tau$, (b) $\theta = 2\pi\tau + \varepsilon$

For $x \in \mathbb{R}$, let

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} = [a_0, a_1, a_2, \cdots],\ a_0 \in \mathbb{Z},\ a_i \in \mathbb{Z}_+,\ i \geq 1$$

be a continued fraction expansion of x.

Define the sequence

$$\{p_i\}_{i \geq 0},\ \{q_i\}_{i \geq 0},\ \{p_{i,k}\}_{i \geq 0, 0<k<a_{i+1}},\ \{q_{i,k}\}_{i \geq 0, 0<k<a_{i+1}}$$

by $p_0 = a_0$**,** $q_0 = 1$**;** $p_1 = a_0 a_1 + 1$**,** $q_1 = a_1$**;**
$p_i = p_{i-2} + a_i p_{i-1}$**,** $q_i = q_{i-2} + a_i q_{i-1}$**,** $i \geq 2$**; and**
$p_{i,k} = p_{i-1} + k p_i$**,** $q_{i,k} = q_{i-1} + k q_i$**,** $i \geq 0$**,** $0 < k < a_{i+1}$**.**

$\frac{p_i}{q_i} = [a_0, a_1, \cdots, a_i]$ **is called a principal convergent of x and**

$\frac{p_{i,k}}{q_{i,k}} = [a_0, a_1, \cdots, a_i, k]$ **is called an intermediate convergent of x.**

Lemma 5

Let $0 < x < 1$**. Let** m, n, a, b **be positive integers such that**

$$\frac{a}{m} < x < \frac{b}{n}, \quad bm - an = 1.$$

Then $\frac{a}{m}$**,** $\frac{b}{n}$ **are principal or intermediate convergents of** x**, at least one of which is principal.**

Let $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{D} \backslash \mathbb{R}$. Let $m, n > 0$ be relatively prime integers.

Let a, b be integers such that $bm - an = 1$.

Proposition 6

Suppose that $\zeta^m, \zeta^n \notin \mathbb{R}_-$.

Suppose that $\mathcal{T} = \{T_j := \square \mathrm{A}_j \mathrm{A}_{j+m} \mathrm{A}_{j+m+n} \mathrm{A}_{j+n}\}_{j \in \mathbb{Z}}$ is a quadrilateral spiral multiple tiling of a multiplicity $|v|$ and (m, n) is an opposed parastichy pair.

Then $\dfrac{a}{m}$, $\dfrac{b}{n}$ are principal or intermediate convergents of $\dfrac{1}{v}\left(\dfrac{\theta}{2\pi} + \ell\right)$, at least one of which is principal,

where ℓ is given by $\ell = a[[\dfrac{n\theta}{2\pi}]] - b[[\dfrac{m\theta}{2\pi}]]$.

Proof of proposition 6:

We may suppose that $\arg(\zeta^n) < 0 < \arg(\zeta^m)$ without loss of generality. So we have

$$\langle\frac{n\theta}{2\pi}\rangle = \frac{n\theta}{2\pi} - [[\frac{n\theta}{2\pi}]] < 0 < \frac{m\theta}{2\pi} - [[\frac{m\theta}{2\pi}]] = \langle\frac{m\theta}{2\pi}\rangle$$

By $\ell = a[[\frac{n\theta}{2\pi}]] - b[[\frac{m\theta}{2\pi}]]$ and $v = m[[\frac{n\theta}{2\pi}]] - n[[\frac{m\theta}{2\pi}]]$, We have

$$[[\frac{m\theta}{2\pi}]] = av - m\ell,\ \ [[\frac{n\theta}{2\pi}]] = bv - n\ell.$$

Thus we obtain

$$n\left(\frac{\theta}{2\pi} + \ell\right) - bv < 0 < m\left(\frac{\theta}{2\pi} + \ell\right) - av,$$

and hence

$$\frac{a}{m} < \frac{1}{v}\left(\frac{\theta}{2\pi} + \ell\right) < \frac{b}{n},\ \ bm - an = 1.$$

□

Triangular spiral multiple tilings

Let

$$\mathcal{T} = \{T_j := \square \mathrm{A}_j \mathrm{A}_{j+m} \mathrm{A}_{j+m+n} \mathrm{A}_{j+n}\}_{j \in \mathbb{Z}}$$

be a quadrilateral spiral multiple tiling of multiplicity $|v|$.

If three vertices of a quadrilateral $T_0 = \square \mathrm{A}_0 \mathrm{A}_m \mathrm{A}_{m+n} \mathrm{A}_n$ lie on a same line, then $\mathcal{T}$ becomes a triangular spiral multiple tiling.

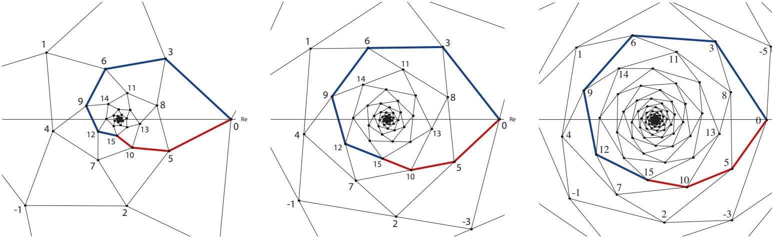


Figure: **Deformation between quadrilateral tilings through a triangular tiling with an opposed parastichy pair $(3, 5)$**

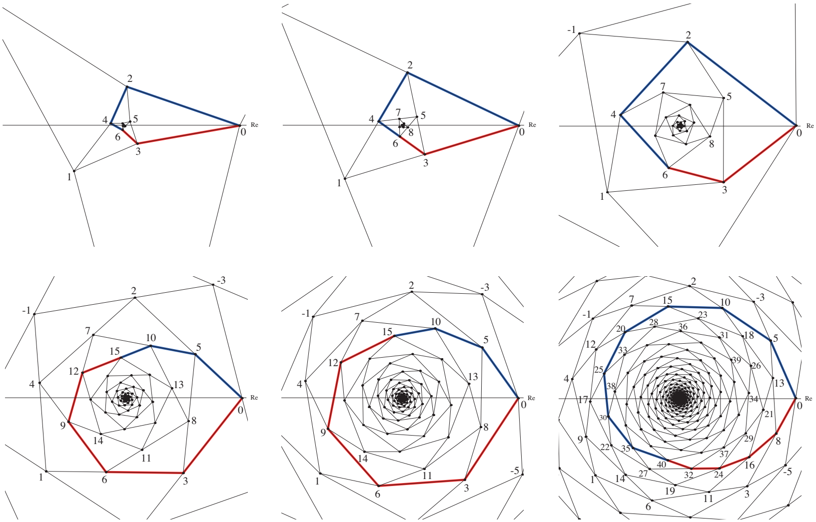


Figure: **Parastichy transition:** $\theta = 2\pi\tau$, $\tau = \frac{1+\sqrt{5}}{2}$, $(2,3) \to (5,3) \to (5,8)$

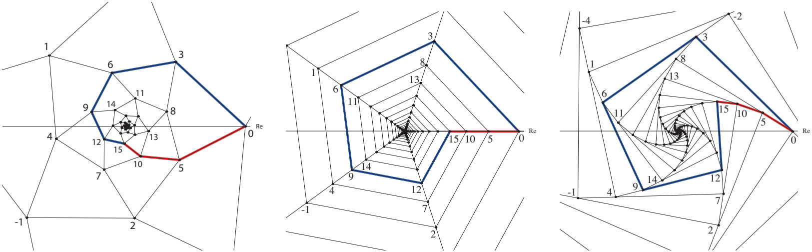


Figure: **Deformation between quadrilateral tilings through a triangular tiling with a non-opposed parastichy pair $(3, 5)$**

Questions

(i) For each pair (m, n), which complex numbers $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{D}\backslash\mathbb{R}$ produce triangular spiral multiple tilings ?

(ii) Which triangles admit spiral multiple tilings ?

Let

$$\phi_{m,k}(z) = \frac{z^k - 1}{z^m - 1} \qquad (3)$$

be a rational function with one complex variable.

Lemma 7

Let $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{C}\backslash\mathbb{R}$**. Let** $m, n > 0$ **be relatively prime integers. Suppose that** $\zeta^m \neq 1$**. Then the following conditions are mutually equivalent.**

(i) **The three points** A_m**,** A_{m+n} **and** A_n **lie on a same line.**

(ii) **The four points** O**,** A_0**,** A_m **and** A_n **lie on a same circle.**

(iii) $r^m \sin n\theta - r^n \sin m\theta + \sin{(m-n)\theta} = 0$**.**

(iv) $\phi_{m,m-n}(\zeta) \in \mathbb{R}$**.**

Main theorems for triangular spiral multiple tilings

We proved the following theorem about triangular spiral multiple tilings with opposed parastichy pairs.

Theorem A

Let $m, n > 0$ **be relatively prime integers.**
Let v **be an integer which satisfies** $0 < |v| < \frac{\max\{m,n\}}{2}$**.**

Let $P_{m,n,v}$ **be the set of generators** $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{D} \backslash \mathbb{R}$
for triangular spiral multiple tilings of multiplicity $|v|$
with an opposed parastichy pair (m, n)**.**

Then $P_{m,n,v}$ **is a branch of a real algebraic curve parameterized by** $\theta = \arg(\zeta)$**.**

Moreover, the union $P = \bigcup_{v>0} \bigcup_{(m,n)} P_{m,n,v}$ **is a dense subset of** $\mathbb{D}$**.**

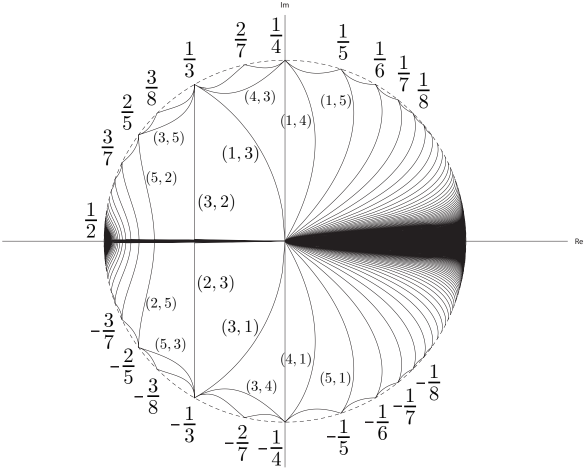


Figure: **The set of generators for triangular spiral tilings of multiplicity $v = 1$, with opposed parastichy pairs (m, n): A rational number x on the unit circle denotes $e^{2\pi\sqrt{-1}x}$**

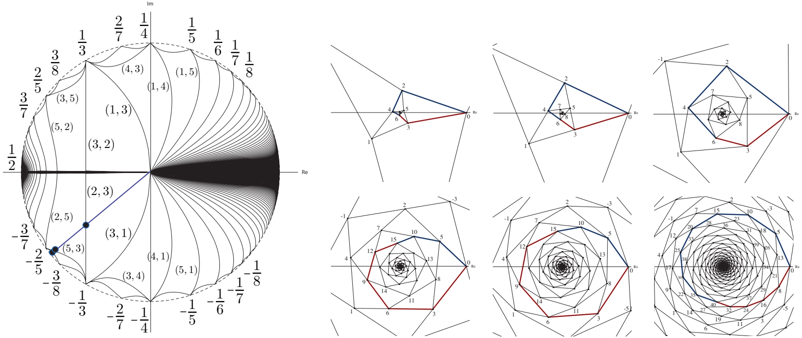


Figure: **Parastichy transition:** $\theta = 2\pi\tau$, $\tau = \frac{1+\sqrt{5}}{2}$, $(2,3) \to (5,3) \to (5,8)$

■ **Principal convergents of the Golden section $\tau = \frac{1+\sqrt{5}}{2}$:**

$$\frac{1}{1}, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, \cdots, \tau, \cdots, \frac{34}{21}, \frac{13}{8}, \frac{5}{3}, \frac{2}{1}$$

■ **Parastichy transition:**

$$(2,3) \to (5,3) \to (5,8) \to (13,8) \to (13,21), \cdots.$$

Next, we proved the following theorem about triangular spiral multiple tilings with non-opposed parastichy pairs.

Theorem B

Let $m, n > 0$ be relatively prime integers.
Let v be an integer which satisfies $0 < |v| < \frac{n}{2}$.

Let $Q_{m,n,v}$ be the set of generators $\zeta = re^{\sqrt{-1}\theta} \in \mathbb{D} \setminus \mathbb{R}$ for triangular spiral multiple tilings of multiplicity $|v|$ with a non-opposed parastichy pair (m, n).

Then $Q_{m,n,v}$ is a branch of a real algebraic curve parameterized by $r = |\zeta|$.

Moreover, the union $Q_v = \bigcup_{(m,n)} Q_{m,n,v}$ is a dense subset of $\mathbb{D}$.

Triangles which admit spiral multiple tilings

Let $\arg\zeta^m = -\arg\zeta^n = \pi/3$**, and let** $v = 1$ **.**

$n\arg\zeta^m - m\arg\zeta^n = 2\pi\cdot 1 \Rightarrow m+n=6$. $(m,n)=(1,5)$, $\theta=\pi/3$.

$r^m \sin n\theta - r^n \sin m\theta + \sin(m-n)\theta = 0$ **has a unique root**

$0 < r = 0.7548\cdots < 1$.

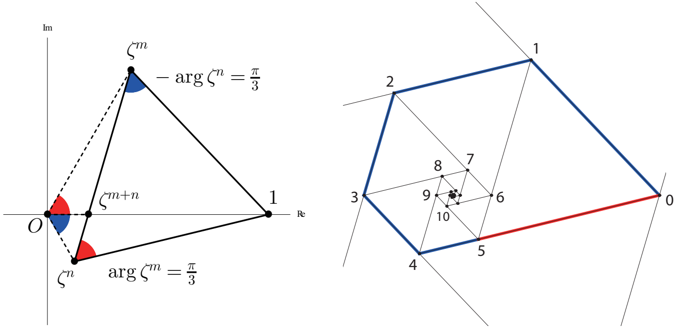


Figure: **A spiral tiling by equilateral triangles**

Let $\arg\zeta^m = \arg\zeta^n = \pi/3$**, and let** $v = 1$**.**

$n\arg\zeta^m - m\arg\zeta^n = 2\pi v \Rightarrow -m + n = 6$. $(m, n) = (1, 7)$, $\theta = \pi/3$.

However, $r^m \sin(m+n)\theta - r^{m+n}\sin m\theta - \sin n\theta = 0$ **doesn't have a solution** $0 < r < 1$**.**

Hence we can't obtain a spiral tiling by equilateral triangles, with an non-opposed parastichy pair.

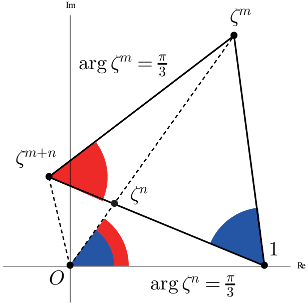


Figure: **A equilateral triangle of the case of non-opposed parastichy pair**

Let $\arg\zeta^m = \pi/3$, $\arg\zeta^n = -\pi/6$, **and let** $v = 1$.

$n\arg\zeta^m - m\arg\zeta^n = 2\pi v \Rightarrow m + 2n = 12$

$(m, n) = (2, 5)$, $\theta = -5\pi/6$.

The equation $r^m \sin n\theta - r^n \sin m\theta + \sin(m-n)\theta = 0$

has a unique root $0 < r = 0.9214\cdots < 1$.

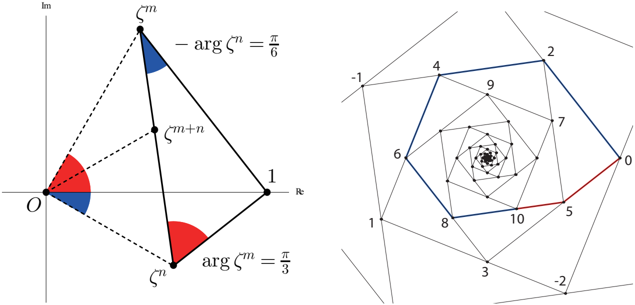


Figure: **A spiral tiling by right triangles with angles** 30°, 60°, 90°

Let $\arg\zeta^m = \pi/3$, $\arg\zeta^n = \pi/6$, **and let** $v = 1$

$n\arg\zeta^m - m\arg\zeta^n = 2\pi v \Rightarrow -m + 2n = 12$

$(m,n) = (2,7)$, $\theta = -5\pi/6$.

The equation $r^m \sin(m+n)\theta - r^{m+n}\sin m\theta - \sin n\theta = 0$

has two solutions $r = 0.883\cdots$ **and** $0.754\cdots$.

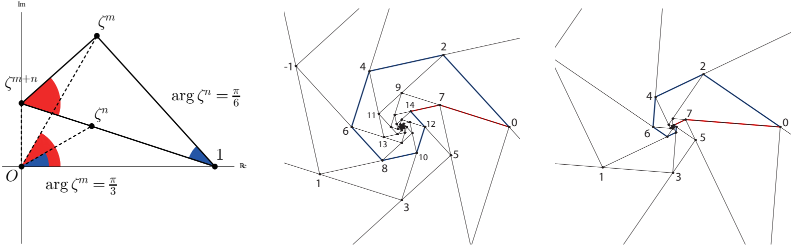


Figure: **Spiral tilings by right triangles with angles** 30°, 60°, 90°

Origami for triangular spiral tilings

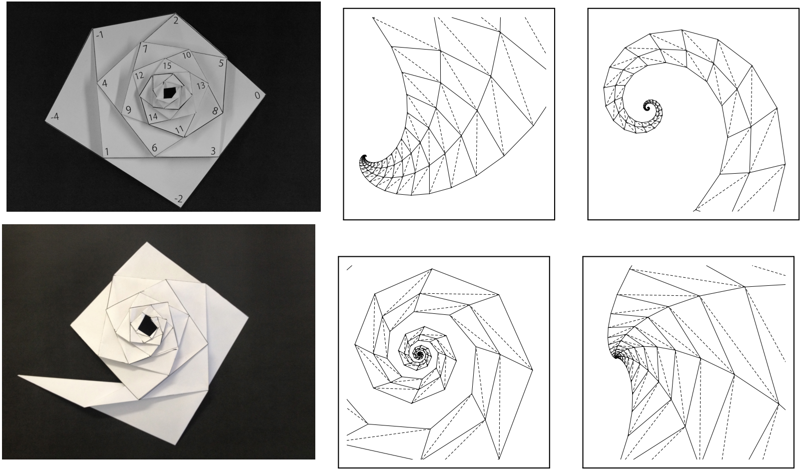


Figure: **Origami sheets for a Fibonacci tornado:**
Solid lines are mountain fold and dashed lines are valley fold.

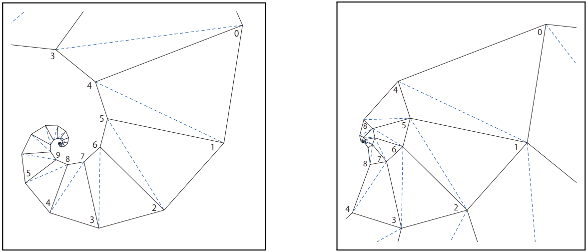


Figure: **Origami sheets for a spiral tiling by regular triangles: Solid lines are mountain fold and dashed lines are valley fold.**

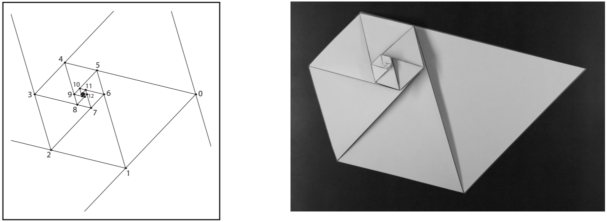


Figure: **Top-down view of an origami of a spiral tiling by regular triangles.**

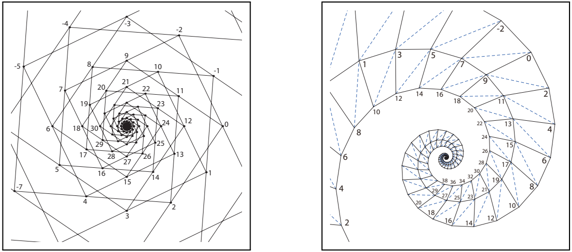


Figure: **An origami sheet for a spiral multiple tiling by right triangles with angles 30°, 60°, 90°**

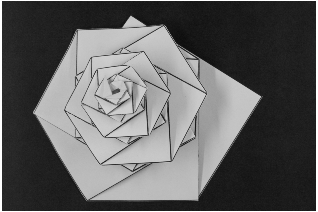


Figure: **Top-down view**

In the right figure,
solid lines are mountain fold and
dashed lines are valley fold.

These figures were chosen the cover page of Journal of Physics A: Mathematical and Theoretical. 45. 23, 2012.

Thank you for your attention...